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Computation Techniques and Computer Programs for Use in Electromagnetics, A Survey

Abstract

This document describes four different computation techniques for calculations of the electromagnetic fields in a complex region, i.e. to solve Maxwell's equations. The techniques described are : TLM (Transmission-Line Modelling), MoM (Method of Moments), FEM (Finite Element Method) and FDM (Finite Difference Method).

All of these techniques are usable from low frequencies up to and just above the resonance frequency of the structure analysed. Besides the mathematical foundation of the computation techniques, various computer programs, commercially available, are discussed.

Key words : FDM, FDTD, Finite Difference Method, FEM, Finite Element Method, MoM, Method of Moments, Numerical methods, TLM, Transmission-Line Matrix, Transmission-Line Modelling

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Summary

This document describes four different numerical computation techniques usable for calculations of the electromagnetic field in a complex region. The techniques are usable from low frequencies up to and just above the resonance frequency of the analysed structure. This limitation is due to the limited storage capacity of the computers of today and the fact that increased complexity of the analysed structure will significantly increase the computation time.

The techniques described are : TLM (Transmission-Line Modelling), MoM (Method of Moments), FEM (Finite Element Method) and FDM (Finite Difference Method).

The TLM is based on the fact that there are certain similarities between the equations obtained from transmission-line theory and the equations obtained directly from Maxwell's equations. Thus, it can be shown that a transmission-line mesh can be used for simulation of wave propagation in a medium. By solving the set of transmission-line equations for the voltages and currents in the mesh and using the analogies with Maxwell's equations, conclusions about wave propagation can be drawn.

The MoM is a method to reduce a functional equation to a matrix equation. In electromagnetics this technique is often used in order to reduce an integral equation to a matrix equation. The matrix can easily be inverted on a computer in order to give, for instance, the wanted source distribution.

The FEM is a method closely related to MoM but is often used for differential equations instead of integral equations. When used for differential equations the method is suitable for finite bounded regions.

The last method described is the FDM which is often used in the time domain. When used for time domain problems, the method is a time and space stepping method. The stepping is performed in an iterative manner and is continued until a steady state is achieved.

Besides the mathematical foundation of the computation techniques, a brief description of selected computer codes, commercially available, are given.

1 Introduction

With the computer techniques of today, it is possible to perform advanced computations with many variables in a reasonably short time. This fact makes it interesting to use computers for calculations of the electromagnetic fields.

In order to calculate the electromagnetic field in a volume, it is usually necessary to divide the volume into many small sub volumes. Each of those sub volumes represents, in the computer, one or more memory cells. Therefore, when calculating electromagnetic fields for a large volume the memory space in the computer has to be large.

Another limiting factor is the time it will take for computation of the fields. The computation time is dependent of the number of unknown variables, often proportional to the number of unknowns raised to a number greater than two. In turn the number of variables is dependent of the complexity of the system analysed. Depending on what numerical computation technique being employed, the number of iterations required for a given accuracy varies. This means that it is not only the size and complexity of the problem that determine the computation time required; what algorithm being used is also an important factor.

From the discussion above it is clear that it is important to chose the most appropriate computation technique for a given problem in order to achieve good accuracy as well as short computation time. In order to help the reader with this choice this document contains descriptions of four different computation techniques suitable for computer computations. The computation techniques described are TLM, MoM, FEM and FDM. Besides the mathematical foundation of the techniques, commercially available computer codes and their possibilities are discussed.

All of the computation techniques discussed in this document can be used for structures of a total length of up to a few wavelengths. This means that the computation techniques are usable from low frequencies up to and just above the resonance frequency of the analysed structure. However, this is seldom a limitation because this frequency region is often the region of interest in technologies as, for instance, EMC where the problems often are in this range.

2 Computation techniques

When the electromagnetic field in a region is to be calculated, Maxwell's equations in one or another form has to be solved. Since this only can be done analytically for a very limited, and simple, number of structures a numerical approach is desired.

When solving Maxwell's equations two main classes of computation techniques can be employed, those based on integral equations and those based on differential equations. Both techniques can be applied in the time domain or in the frequency domain.

The integral equation formulation can, for instance, be developed from Maxwell's equations using Green's theorem, see [2]. The resulting integral equation will be an integral over the sources of the problem, for instance the current on a conducting body. Thus, as an example, an antenna problem will be formulated as an integral only over the current on the antenna itself. When the problem is formulated as an integral equation a numerical solution technique such as the method of moments (ch. 4) or the finite element method (ch. 5) can be employed.

The differential equation approach is a direct attack on Maxwell's equations given in differential form, as given in chapter 2.1. This formulation is most suitable for a finite region because all of the region has to be included in the calculations. Techniques using this approach includes transmission-line modelling (ch. 3), the finite difference method (ch. 6) and even the finite element method (ch. 5).

2.1 Maxwell's equations

Maxwell's equations in the time-domain are :

$$\nabla \times \mathcal{E} = - \frac{\partial \mathcal{B}}{\partial t} \quad (2:1)$$

$$\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J} \quad (2:2)$$

$$\nabla \cdot \mathcal{B} = 0 \quad (2:3)$$

$$\nabla \cdot \mathcal{D} = q_v \quad (2:4)$$

And if the matter is linear, in the simple sense, the following relationships are valid :

$$\mathcal{D} = \epsilon \mathcal{E} \quad (2:5)$$

$$\mathcal{B} = \mu \mathcal{H} \quad (2:6)$$

$$\mathcal{J} = \sigma \mathcal{E} \quad (2:7)$$

Where :

- \mathcal{E} = electric intensity, [V/m]
- \mathcal{H} = magnetic intensity, [A/m]
- \mathcal{D} = electric flux density, [C/m²]
- \mathcal{B} = magnetic flux density, [Wb/m²]
- \mathcal{J} = electric current density, [A/m²]
- q_v = electric charge density, [C/m³]
- ϵ = capacitivy of the medium, [F/m]
- μ = inductivity of the medium, [H/m]
- σ = conductivity of the medium, [S/m]

For the time-harmonic case, i.e. when the relation between the instantaneous quantities and the complex quantities can be written as $\mathcal{E} = \sqrt{2} \operatorname{Re} (\mathbf{E}e^{j\omega t})$, Maxwell's equations can be written :

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \quad (2:8)$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J} \quad (2:9)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2:10)$$

$$\nabla \cdot \mathbf{D} = q_v \quad (2:11)$$

Where : $\omega = 2\pi f =$ angular frequency, [rad/s]

$$\mathbf{E} = |\mathbf{E}|e^{j\alpha}$$

$|\mathbf{E}| =$ Root-mean-square value

And the constitutive relationships, equation (2:5) - (2:7) with the instantaneous quantities replaced with the complex ones, remain the same. These are Maxwell's equations in the frequency domain.

If the source terms in Maxwell's equations, i.e. \mathbf{J} and q_v , are removed the equations are found to be valid even if \mathbf{E} and \mathbf{H} are exchanged with their dual quantities. The dual quantities are :

$$\mathbf{E}' = \pm \sqrt{\frac{\mu}{\epsilon}} \mathbf{H}$$

$$\mathbf{H}' = \mp \sqrt{\frac{\epsilon}{\mu}} \mathbf{E}$$

Since the dual quantity for the electric field is proportional to the magnetic field, and vice versa, this property can be very useful in many cases. For instance if a problem for the magnetic field is solved, the dual problem for the electric field can be solved by simply exchanging symbols.

In order to make Maxwell's equations dual also when sources are present, a magnetic current, \mathbf{M} , and a magnetic charge density, q_m , can be introduced. Using the relations given in equations (2:5) - (2:7) Maxwell's equations can now be written as :

$$\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} - \mathbf{M} \quad (2:12)$$

$$\nabla \times \mathbf{H} = j\omega \epsilon \mathbf{E} + \mathbf{J} \quad (2:13)$$

$$\nabla \cdot \mathbf{H} = q_m / \mu \quad (2:14)$$

$$\nabla \cdot \mathbf{E} = q_v / \epsilon \quad (2:15)$$

Equation (2:12) becomes equal to (2:13) and equation (2:14) becomes equal to equation (2:15) if the following interchanges are performed :

$$\mathbf{E} \leftrightarrow \mathbf{H} \quad \epsilon \leftrightarrow -\mu \quad \mathbf{J} \leftrightarrow -\mathbf{M} \quad q_v \leftrightarrow -q_m$$

or :

$$\mathbf{E} \rightarrow \mathbf{H} \quad \mathbf{H} \rightarrow -\mathbf{E} \quad \epsilon \rightarrow \mu \quad \mathbf{J} \rightarrow \mathbf{M}$$

$$q_v \rightarrow q_m \quad \mathbf{M} \rightarrow -\mathbf{J} \quad q_m \rightarrow -q_v \quad \mu \rightarrow \epsilon$$

Instead of solving Maxwell's equations directly for E or H it is often more convenient to take a "detour" by using so-called auxiliary quantities. Several possible auxiliary quantities exist, the ones treated here are the electric and magnetic vector potentials.

By introducing the magnetic vector potential, \mathbf{A} , and the electric vector potential, \mathbf{F} , the electric and magnetic fields can be written as, see for instance [4] :

$$\mathbf{E} = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) - \frac{1}{\epsilon}\nabla\times\mathbf{F}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} - j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{F})$$

Where :

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\mu\mathbf{J}$$

$$\nabla^2\mathbf{F} + \omega^2\mu\epsilon\mathbf{F} = -\epsilon\mathbf{M}$$

Thus, the vector potentials satisfy the inhomogeneous vector Helmholtz equation. It should also be noted that if the co-ordinate system is the rectangular one, the components of the vector potentials satisfy the scalar inhomogeneous Helmholtz equation. It is therefore often sufficient to study the solution to the scalar Helmholtz equation in order to solve a complex field problem.

3 Transmission-Line Modelling, TLM

The transmission-line modelling, also known as the transmission-line matrix method, is a numerical technique for solving field problems using circuit equivalents. The technique is based on the fact that the equations obtained from the transmission-line theory have certain similarities with the equations obtained directly from Maxwell's equations.

The region under consideration is modelled by a large number of short transmission-line sections connected together in a mesh. The next step is to solve the resulting set of transmission-line equations for the voltages and currents at all nodes in the mesh. This is done in the time domain with an iterative method. As discussed below the voltages and currents can be thought as representing the wanted electric and magnetic fields.

3.1 The transmission-line section

Usually the equivalent circuit model for the short transmission-line section is chosen to be the T-type section shown in figure 3:1. Other types of equivalent circuits are possible, but the resulting set of equations will be the same.

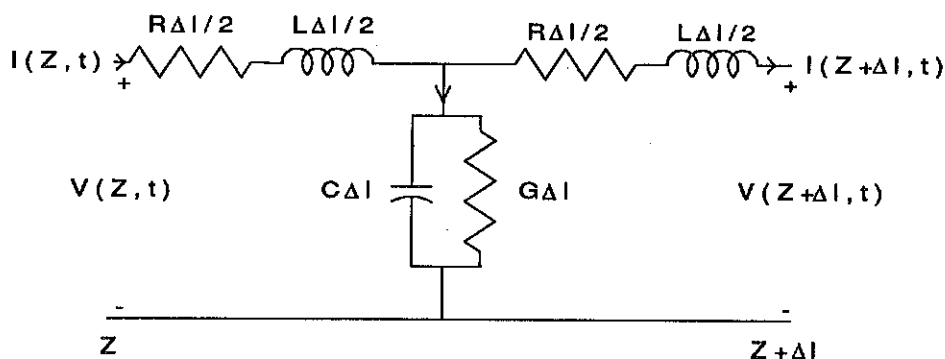


Figure 3:1. T-type equivalent circuit model for a short section of a two conductor transmission-line.

In figure 3:1 the line parameters R , L , G and C are resistance per unit length, inductance per unit length, conductance per unit length and capacitance per unit length of the line respectively. In the derivation of equations that can be used for representation of wave propagation in a lossless media, the transmission-line parameters that represent losses can be ignored, i.e. $R=G=0$ in figure 3:1. The equivalent circuit in figure 3:1 then reduces to the one shown in figure 3:2.

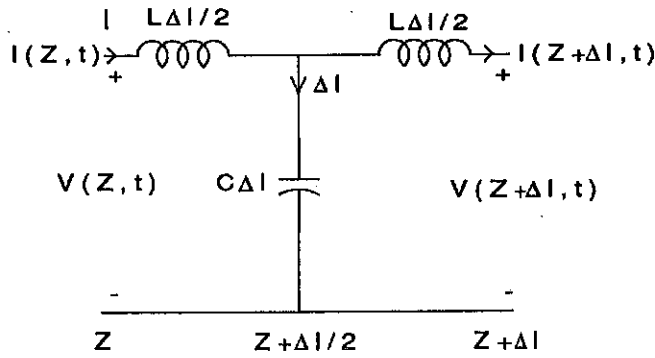


Figure 3:2. T-type equivalent circuit model for a short section of a two conductor transmission-line used to derive the wave equation.

3.2 Equivalence between TLM and Maxwell's equations

In order to show how the transmission-line modelling can be used for wave propagation problems, the equivalence between the transmission-line equations and Maxwell's equations for the two-dimensional case is shown. The following discussion is valid for wave propagation in a lossless media. The mesh node obtained by the connection of two perpendicular transmission-line sections is shown in figure 3:3.

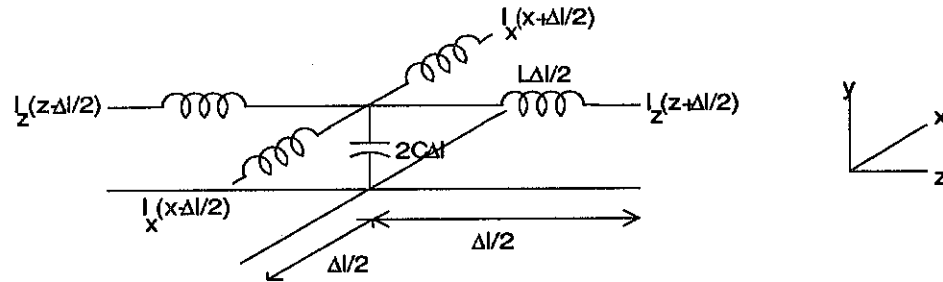


Figure 3:3. Equivalent circuit for a node in the mesh.

Applying Kirchoff's current law to the node in figure 3:3 gives :

$$I_x(x-\Delta l/2) - I_x(x+\Delta l/2) + I_z(z-\Delta l/2) - I_z(z+\Delta l/2) = 2C\Delta l \frac{\partial V_y}{\partial t} \quad (3:1)$$

Dividing both sides with Δl and taking the limit as $\Delta l \rightarrow 0$ gives :

$$-\frac{\partial I_z}{\partial z} - \frac{\partial I_x}{\partial x} = 2C \frac{\partial V_y}{\partial t} \quad (3:2)$$

Applying Kirchoff's voltage law around the loop in the x-y plane in figure 3:3 gives :

$$V_y(x-\Delta l/2) - L\Delta l/2 \frac{\partial I_x(x-\Delta l/2)}{\partial t} - L\Delta l/2 \frac{\partial I_x(x+\Delta l/2)}{\partial t} - V_y(x+\Delta l/2) = 0 \quad (3:3)$$

Dividing both sides with Δl and taking the limit as $\Delta l \rightarrow 0$ gives :

$$\frac{\partial V_y}{\partial x} = -L \frac{\partial I_x}{\partial t} \quad (3:4)$$

The same procedure for the loop in the y-z plane gives :

$$\frac{\partial V_y}{\partial z} = -L \frac{\partial I_z}{\partial t} \quad (3:5)$$

Differentiating equation (3:2) with respect to t, equation (3:4) with respect to x and equation (3:5) with respect to z gives the following equations :

$$-\frac{\partial^2 I_z}{\partial z \partial t} - \frac{\partial^2 I_x}{\partial x \partial t} = 2C \frac{\partial^2 V_y}{\partial t^2} \quad (3:6)$$

$$\frac{\partial^2 V_y}{\partial x^2} = -L \frac{\partial^2 I_x}{\partial t \partial x} \quad (3:7)$$

$$\frac{\partial^2 V_y}{\partial z^2} = -L \frac{\partial^2 I_z}{\partial t \partial z} \quad (3:8)$$

Insertion of equation (3:7) and (3:8) into equation (3:6) finally gives :

$$\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2} = 2LC \frac{\partial^2 V_y}{\partial t^2} \quad (3:9)$$

which is Helmholtz equation in two-dimensional space. The solution to equation (3:9) can be found by using the method of separation of variables. The solution will be in the form of a summation of harmonic functions.

In order to show the similarity between this equation and the wave equation for two-dimensional space derived from Maxwell's equations, the following steps are taken :

Maxwell's equations for a source free region are, see chapter 2.1 :

$$\nabla \times \mathcal{E} = -\mu \frac{\partial \mathcal{H}}{\partial t} \quad (3:10)$$

$$\nabla \times \mathcal{H} = \epsilon \frac{\partial \mathcal{E}}{\partial t} \quad (3:11)$$

Expansion of equation (3:10) and (3:11) in the rectangular co-ordinate system gives the following set of equations :

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial \mathcal{H}_x}{\partial t} \quad (3:12a)$$

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\mu \frac{\partial \mathcal{H}_y}{\partial t} \quad (3:12b)$$

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\mu \frac{\partial \mathcal{H}_z}{\partial t} \quad (3:12c)$$

$$\frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} = \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \quad (3:12d)$$

$$\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} = \epsilon \frac{\partial \mathcal{E}_y}{\partial t} \quad (3:12e)$$

$$\frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \quad (3:12f)$$

Considering the case for which $\mathcal{E}_x = \mathcal{E}_z = \mathcal{H}_y = 0$ and all derivatives with respect to y equals zero (no variation in the y -direction). This is a TM to y mode for the special case when all derivatives with respect to y equals zero. For this special case the set of equations (3:12) reduces to :

$$\frac{\partial \mathcal{E}_y}{\partial z} = \mu \frac{\partial \mathcal{H}_x}{\partial t} \quad (3:13)$$

$$\frac{\partial \mathcal{E}_y}{\partial x} = -\mu \frac{\partial \mathcal{H}_z}{\partial t} \quad (3:14)$$

$$\frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} = \epsilon \frac{\partial \mathcal{E}_y}{\partial t} \quad (3:15)$$

Differentiating equation (3:13) with respect to z , equation (3:14) with respect to x , equation (3:15) with respect to t and substituting the resulting equations gives another Helmholtz equation :

$$\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2} = \mu \epsilon \frac{\partial^2 \mathcal{E}_y}{\partial t^2} \quad (3:16)$$

Comparing equations (3:13) - (3:16) with the equations derived from the transmission-line equations, i.e. equations (3:2),(3:4),(3:5) and (3:9), gives the following equalities :

$$\mathcal{E}_y = V_y \quad (3:17a)$$

$$\mathcal{H}_x = -I_z \quad (3:17b)$$

$$\mathcal{H}_z = I_x \quad (3:17c)$$

$$\mu = L \quad (3:17d)$$

$$\epsilon = 2C \quad (3:17e)$$

So, it is evident from the equalities (3:17) that the transmission-line matrix can be used as a model for solving wave propagation problems. In the equivalent circuit the node voltage V_y represents the electric field component \mathcal{E}_y and the currents I_x and $-I_z$ represent the magnetic fields \mathcal{H}_z and \mathcal{H}_x respectively. The inductance per unit length represents the inductivity of the medium and twice the capacitance represents the capacitivity of the medium.

The equalities derived above is not the only ones possible. The dual nature of electric and magnetic fields makes it possible to simulate, for example, the TE to y mode by the following equalities :

$$\mathcal{H}_y = V_y$$

$$\mathcal{E}_x = I_z$$

$$\mathcal{E}_z = -I_x$$

$$\mu = 2C$$

$$\epsilon = L$$

3.3 Wave properties of the mesh

Independent of which relationship between field and network variables is used the wave properties of the mesh remain the same.

If the voltage and current waves on the Transmission-line section in figure 3:2 propagates at the speed of light, c , it follows that the inductance and capacitance per unit length are related by :

$$c = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{LC}} \quad (3:18)$$

By comparing equation (3:18) with the equalities (3:17) it is noticed that the propagation velocity in the TLM mesh, figure 3:3, is $1/\sqrt{2}$ of the speed of light, c . Thus, using the equalities (3:17) the wave propagation speed in the mesh is a factor $1/\sqrt{2}$ times the wave propagation speed in the simulated medium. This wave propagation speed in the mesh is valid for low frequencies, i.e. when the length of the transmission-line section, Δl , is short compared with the wavelength. When the frequency is increased, the propagation along one of the mesh axis has to be treated as propagation along a periodic structure. Performing such a treatment, the following dispersion relation for propagation along the main mesh axes can be shown, [7] (axial propagation in figure 3:4) :

$$\sin(\beta_n \Delta l / 2) = \sqrt{2} \sin[\omega \Delta l / (2c)] \quad (3:19)$$

Where : β_n = propagation constant in the network.

The resulting ratio of velocities in the TLM and in free space, u_n/c , is shown in figure 3:4.

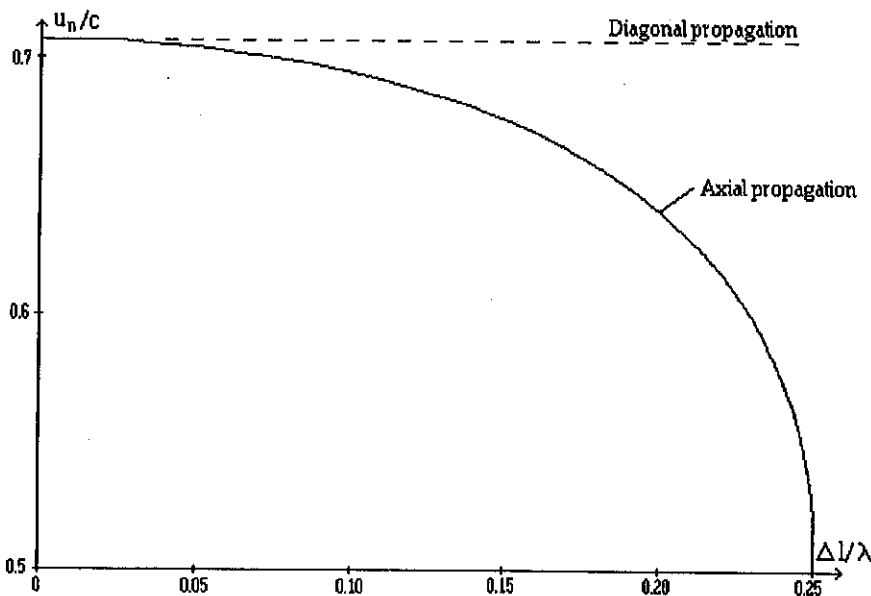


Figure 3:4. Dispersion of the propagation velocity in a two-dimensional TLM network.

From figure 3:4 it can be concluded that the TLM network can only represent Maxwell's equations for frequencies below the network cut-off, which occurs at $\Delta l / \lambda = 1/4$.

It can also be concluded that the TLM network simulates an isotropic propagation medium only as long as all frequencies are well below the cut-off frequency. For such a case the propagation velocity may be considered constant and equal to $c/\sqrt{2}$ (for two dimensions).

Another important parameter of the TLM network is the characteristic impedance in the mesh. This parameter has to be known when boundary conditions are considered (ch. 3.5).

The free space intrinsic impedance is : $\eta = \sqrt{\frac{\mu}{\epsilon}} \approx 377 \Omega$

Using the relations (3:17) the free space intrinsic impedance can be expressed, using the network parameters, as :

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{L}{2C}}$$

The impedance of the Transmission-line section, figure 3:2, can be written as :

$$\sqrt{\frac{L}{C}} = \eta\sqrt{2}$$

Thus, the intrinsic impedance of the line is equal to the impedance of the simulated (free space) medium times the square-root of two.

3.4 Basic algorithm

If a voltage impulse is launched into terminal 1 of the node in figure 3:5 one fourth of the energy will be launched into terminal 2, 3, 4 and one fourth of the energy will be reflected back into terminal 1. If the incident voltage impulse is of unit magnitude, the transmitted and reflected impulses will all have a voltage magnitude of one half.

For the more general case of four impulses being incident at time $t=k\Delta l/c$ on the four terminals of a node, the total voltage impulse reflected along line n at time $t=(k+1)\Delta l/c$ will be (by superposition) :

$${}_{k+1}V_n^r = \frac{1}{2} \sum_{m=1}^4 {}_kV_m^i - {}_kV_n^i \quad (3:20)$$

Where : "i" and "r" denotes incident and reflected respectively

Since any impulse emerging from a node (reflected impulse) at position (x,z) automatically becomes an incident impulse on the neighbouring node, the following relationships can be determined :

$${}_{k+1}V_1^i(x,z) = {}_{k+1}V_3^r(x-\Delta x,z) \tag{3:21a}$$

$${}_{k+1}V_2^i(x,z) = {}_{k+1}V_4^r(x,z-\Delta z) \tag{3:21b}$$

$${}_{k+1}V_3^i(x,z) = {}_{k+1}V_1^r(x+\Delta x,z) \tag{3:21c}$$

$${}_{k+1}V_4^i(x,z) = {}_{k+1}V_2^r(x,z+\Delta z) \tag{3:21d}$$

Equation (3:20) and (3:21) states that if magnitudes, positions and directions of all impulses are known at time $t=k\Delta l/c$ the corresponding values at time $t=(k+1)\Delta l/c$ can be obtained. Thus, the impulse response of the network can be found by initially fixing the state of the network at time $t=0$ and then calculating the state of the network at successive time intervals $(k=1, 2, 3, \dots)$. This procedure is shown in figure 3:6, where the excitation is a unit Dirac impulse at one node.

From the above it also follows that any node (one or many) in the network can serve as input and/or output.

The impulse response can be used for calculation of the network response for any input wave form. In the time domain this is accomplished by convolution of the impulse response with the input wave form. If the response to a sinusoidal excitation is wanted the Fourier transform can be used remembering that the answer is valid only for frequencies well below the cut-off frequency, see chapter 3.3.

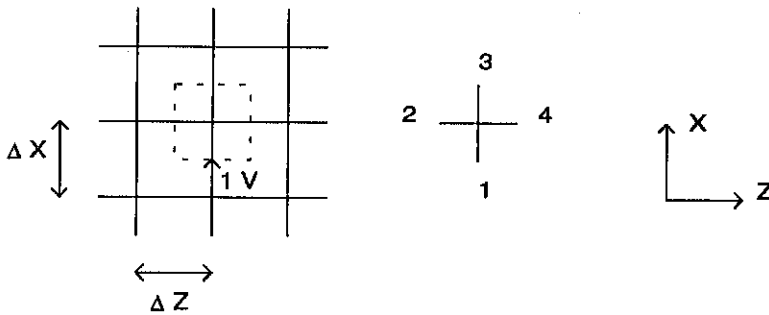


Figure 3:5. Excitation of a node.

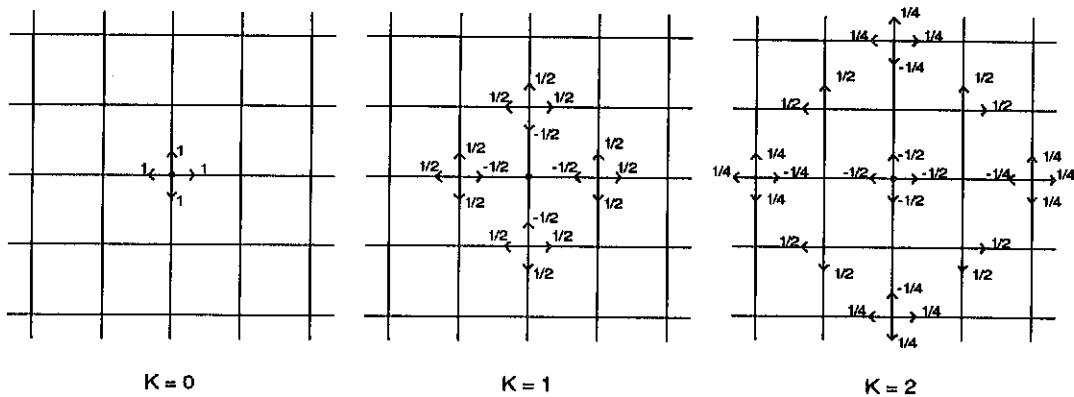


Figure 3:6. Iteration procedure for a unit Dirac voltage impulse excitation, $t = K\Delta l/c$.

3.5 Boundary representation

Boundaries are represented by defining the reflection coefficient at the appropriate positions in the network. To ensure synchronism the boundaries are placed halfway between two nodes. This is achieved by making the distance between two nodes, Δl , equal to an integer fraction of the structure dimensions.

The reflection coefficient for a wave travelling along a Transmission-line section can be written as (see chapter 3.3) :

$$\rho = \frac{Z - \sqrt{Z_0}}{Z + \sqrt{Z_0}} \quad (3:22)$$

Where : Z is normalised to the impedance of the simulated medium.

The reflection coefficient for a few different types of boundaries are presented in table 3:1.

BOUNDARY TYPE	REFLECTION COEFFICIENT, ρ
Electric wall (short circuit)	-1
Magnetic wall (open circuit)	1
Free space (matched)	≈ -0.1716

Table 3:1. Reflection coefficient for different boundaries.

It should be noted that the values given in table 3:1 are valid if the voltage, V_y , in the transmission-line mesh is representing the electric field. If the voltage, on the other hand, is representing the magnetic field the values for electric and magnetic walls have to be interchanged.

3.6 Expansion to three-dimensions

Expansion of the two-dimensional TLM to three-dimensions can be accomplished by replacing the basic shunt node, figure 3:3, with a hybrid TLM cell. In order to simulate all six field components the hybrid cell must consist of three series nodes and three shunt nodes. The voltages at the three shunt nodes represent the electric field components and the currents at the series nodes represent the magnetic field components.

The wave properties of the three-dimensional TLM are similar to that of the two-dimensional TLM, chapter 3.3, with the exception of the propagation velocity. The low-frequency velocity can be shown to be [8] $c/2$ for the three-dimensional TLM ($c/\sqrt{2}$ for two-dimensions).

3.7 Computer codes

The TLM is the most recently developed method of the four methods discussed in this document, thus the supply of computer codes is limited. Here are two codes briefly described, one for two dimensional problems and one for general three-dimensional problems.

"The electromagnetic wave simulator" is a computer code based on a two-dimensional formulation of TLM. The code has a limited range of use since only two-dimensional problems can be treated. However, the code is easy to use and the propagation of waves can be seen in "real time" on the screen. This makes the code educational and using the code is a good way to become familiar with the TLM. The code, which is developed for PCs, is sold together with a comprehensive documentation by John Wiley software.

The computer code "STRIPES" supplied by Kimberley Communications Consultants Ltd. is based on a general three-dimensional formulation of the TLM. The code computes the time domain impulse response of the analysed system. The code also have a Fourier transformation capability which makes it possible to determine the frequency response. Certain post-calculations can be done, for instance convolution in the time domain which gives the possibility to determine the system response for, in principle, any wave form.

The structure that is about to be analysed is feed into the computer in a CAD-like manner, i.e. the structure is drawn on the screen and the material properties can be defined by giving the material constants. When the computations are done the output data can be presented on the screen or on a plotter in many different ways. At present the code can be run on SUN SPARC stations and SUN 3 workstations. The platform must support Fortran-77 and C, and must have at least 4 Mbytes memory.

4 Method of Moments, MoM

The basic idea of the method of moments, MoM, or the moment method [10] is to reduce a functional equation to a matrix equation. The method is general and can be applied to fields of any type.

The method is usually employed in the frequency domain but can also be used for time domain problems.

4.1 General procedure

Consider the following operator equation :

$$L(f) = g \quad (4:1)$$

Where : L is a linear operator
 g is a known function
 f is the function to be determined

Expanding the wanted function, f , in a series of functions, f_n , gives the following expression for the function f :

$$f = \sum_n^N \alpha_n f_n \quad (4:2)$$

Where the α_n are constants and the functions, f_n , are called basis functions. For the summation (4:2) to exactly represent the function, f , the summation usually has to be infinite ($N \rightarrow \infty$). For an approximate representation of the function, f , the summation is usually finite. Insertion of equation (4:2) in equation (4:1) and using the fact that the operator, L , is linear equation (4:1) becomes :

$$\sum_n^N \alpha_n L(f_n) = g \quad (4:3)$$

Defining a suitable inner product, $\langle f, g \rangle$, which is a scalar and satisfies the following relations :

$$\begin{aligned} \langle f, g \rangle &= \langle g, f \rangle \\ \langle \alpha f + \beta g, h \rangle &= \alpha \langle f, h \rangle + \beta \langle g, h \rangle \\ \langle f^*, f \rangle &> 0 ; \text{ if } f \neq 0 \\ \langle f^*, f \rangle &= 0 ; \text{ if } f = 0 \end{aligned}$$

Defining a set of weighting functions, or testing functions, w_m ($m=1,2,3,\dots,N$) and taking the inner product of both sides of equation (4:3) with each w_m gives :

$$\sum_n^N \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle \quad \text{for } m = 1, 2, 3, \dots, N \quad (4:4)$$

or in matrix form :

$$[I_{mn}] [\alpha_n] = [g_m] \quad (4:5)$$

where the matrices $[I_{mn}]$, $[\alpha_n]$ and $[g_m]$ are defined by :

$$[I_{mn}] = \begin{bmatrix} \langle w_1, Lf_1 \rangle & \langle w_1, Lf_2 \rangle & \dots \\ \langle w_2, Lf_1 \rangle & \langle w_2, Lf_2 \rangle & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

$$[\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \dots \end{bmatrix} \quad [g_m] = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \dots \\ \dots \end{bmatrix}$$

If the matrix $[I_{mn}]$ is non-singular the α_n are given by :

$$[\alpha_n] = [I_{mn}]^{-1} [g_m] \quad (4:6)$$

where the matrix $[I_{mn}]^{-1}$ is the inverse of $[I_{mn}]$. The solution is finally found by inserting the α_n in equation (4:2).

Depending on the choice of the functions f_n and w_m the solution may be exact or approximate. The particular choice $f_n = w_n$ is called Galerkin's method.

In electromagnetics the inner product $\langle f, g \rangle$ is almost always defined as an integral over the product fg . This means that every element in the matrix $[I_{mn}]$ involves at least one integration. The integration associated with the inner product can, however, often be avoided by the choice of Dirac pulses as testing functions, so called point matching.

4.2 Procedure for wire antennas and scatters

A very important class of structures to analyse is the wire. This is because practical structures very often contain wires and the fact that it is fairly easy to model an arbitrary structure as a wire grid.

The difference between wire antennas and wire scatters is primarily the location of the source. If the source is located at the wire, the wire is thought as an antenna and if the source is distant the wire is thought as a scatter.

By considering the wire as a number of short segments connected together the problem can be viewed as a N-port problem. The end points of each of the segments are representing a pair of terminals and the wire is represented by short circuiting (if perfect conducting wire) all terminals. The impedance matrix for the N-port network can be found by applying a current source to each port in turn and calculating the open circuit voltage at all ports. The admittance matrix which is the inverse of the impedance matrix can be used for calculation of the current at each port by matrix multiplication with the excitation voltage matrix. This means that the current distribution of the wire, which is the same as the port currents, can be found for any voltage excitation by matrix multiplication. It should also be noted that the voltage excitation matrix represents, for

the case of an antenna, the voltage excitation and for a scatter the applied field (incident field).

In the following pages the moment method is used for derivation of the impedance matrix for a straight wire.

The electric field produced by the currents on a surface can be written as, chapter 2.1 :

$$\mathbf{E}^s = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\mathbf{A}) \quad (4:7)$$

Where : \mathbf{E}^s = scattered electric field

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{s'} \frac{\mathbf{J}e^{-jkR}}{4\pi R} ds' \quad (4:8)$$

\mathbf{J} = surface current density

R = distance from the source point to the field point

ds' = surface element (primed co-ordinates are denoting source co-ordinates)

For thin wires (radius much smaller than wavelength) the following approximations are often used :

A : current is only directed along the wire axis

B : current is uniformly distributed over the wire circumference

From assumption B it follows that the current can be approximated by a filament of current on the wire axis. With these approximations the surface integral, equation (4:8), is reduced to a line integral as follows :

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{l'} \frac{\mathbf{I}e^{-jkR}}{4\pi R} dl' \quad (4:9)$$

Where : \mathbf{I} is related to the corresponding surface quantity by the perimeter $2\pi a$ (a is the wire radius) and the length element is denoted by dl' .

For simplicity, consider the case when the current is directed along one co-ordinate axis as shown in figure 4:1.

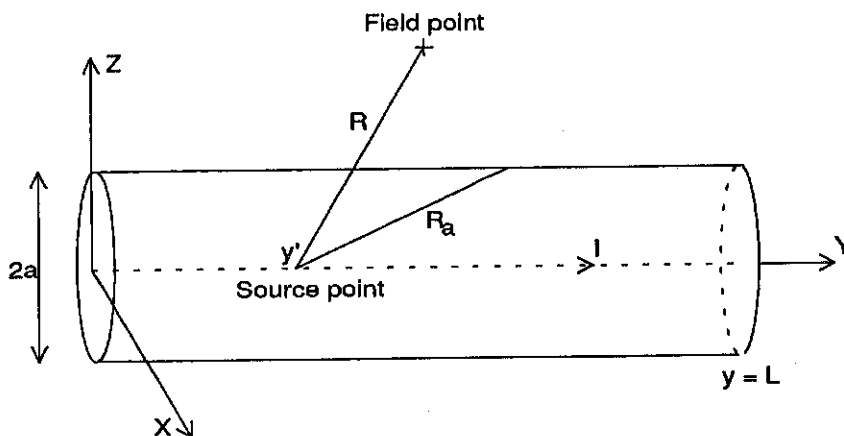


Figure 4:1. Wire geometry.

For the wire in figure 4:1 equation (4:9) becomes :

$$\mathbf{A} = \hat{\mathbf{y}} A = \hat{\mathbf{y}} \frac{\mu}{4\pi} \int_0^L I(y') \frac{e^{-jkR}}{4\pi R} dy' \quad (4:10)$$

And equation (4:7) can be written as :

$$\mathbf{E}^s = \hat{\mathbf{y}} E^s = -\hat{\mathbf{y}} \frac{j\omega\mu}{4\pi} \int_0^L I(y') \left[1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2} \right] \frac{e^{-jkR}}{4\pi R} dy' \quad (4:11)$$

Where : $\hat{\mathbf{y}}$ = unit vector along the wire, see figure 4:1
 $k = \omega\sqrt{\mu\epsilon} = \text{wave number}$
 $L = \text{length of the wire}$
 $R = \text{distance from the source point to the field point}$
 primed co-ordinates are used for source points and unprimed for observation points

The boundary condition at the wire surface is that the total tangential electric fields vanish (perfect conductor). For the case of the thin wire in figure 4:1 this is the same as $\hat{\mathbf{y}} \cdot (\mathbf{E}^i + \mathbf{E}^s) = 0$, where \mathbf{E}^i is the incident electric field.

This means that if the field point in equation (4:11) is assumed to be located on the wire surface the incident field is equal to the negative of the scattered field, $\mathbf{E}^i = -\mathbf{E}^s$. Denoting the distance between the source point on the wire axis and the field point on the wire surface as R_a , the equation for the incident field becomes :

$$\mathbf{E}^i = \hat{\mathbf{y}} E^i = \hat{\mathbf{y}} \frac{j\omega\mu}{4\pi} \int_0^L I(y') \left[1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2} \right] \frac{e^{-jkR_a}}{4\pi R_a} dy' \quad (4:12)$$

Where : $R_a = \sqrt{a^2 + (y-y')^2}$, see figure 4:1

In order to solve the integral equation for the current $I(y')$, equation (4:12), the wire is divided into N segments, each with length $\Delta l = L/N$, see figure 4:2.

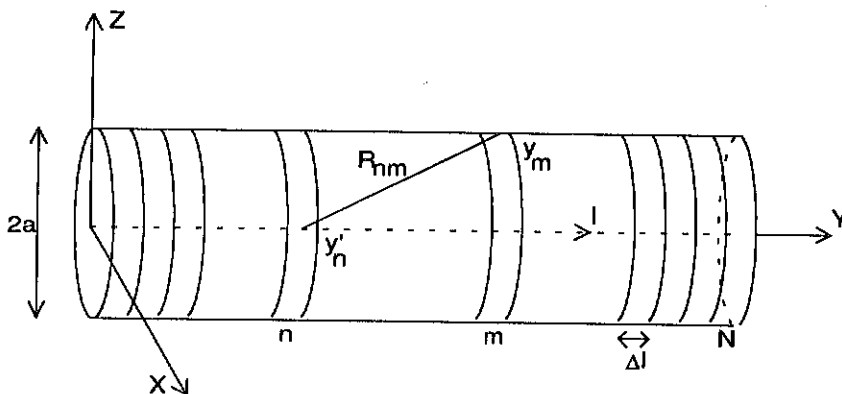


Figure 4:2. Wire segmentation.

Further, the current on each segment is considered as constant, i.e. current on segment number n is equal to α_n . This approximation is good as long as the segments are kept short compared to the variation of the current along the wire.

The current can then be written as :

$$I(y') = \sum_{n=1}^N \alpha_n f_n \quad (4:13)$$

$$\text{Where : } \begin{aligned} f_n &= 1 ; (n-1)\Delta l < y' < n\Delta l \\ f_n &= 0 ; \text{ elsewhere} \\ \alpha_n &= \text{ constants} \end{aligned}$$

So, the basis functions of the moment method are chosen as unit pulses extending only over one segment each. This choice is often referred to as subsectional bases.

Insertion of equation (4:13) in equation (4:12) gives the following integral equation for the coefficients, α_n :

$$E^i = \hat{y} E^i = \hat{y} \frac{j\omega\mu}{4\pi} \sum_{n=1}^N \int_{(n-1)\Delta l}^{n\Delta l} \alpha_n \left[1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2} \right] \frac{e^{-jkR_a}}{4\pi R_a} dy' \quad (4:14)$$

This equation is valid for all y in the range from zero to the total length L . By satisfying this equation only at a limited number of discrete y 's in the range $0 - L$, the equation can be written in a matrix form as follows.

Denoting the points on the wire surface where equation (4:14) is satisfied as y_m and the distance between the source point on the wire axis, y'_n , and the field point, y_m , as R_{nm} (see figure 4:2) equation (4:14) becomes :

$$\sum_{m=1}^N E^i(y_m) = \hat{y} \sum_{m=1}^N E^i(y_m) = \hat{y} \frac{j\omega\mu}{4\pi} \sum_{m=1}^N \sum_{n=1}^N \int_{(n-1)\Delta l}^{n\Delta l} \alpha_n \left[1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2} \right] \frac{e^{-jkR_{nm}}}{4\pi R_{nm}} dy' \quad (4:15)$$

$$\text{Where : } R_{nm} = \sqrt{a^2 + (y_m - y'_n)^2}, \text{ see figure 4:2}$$

The choice of satisfying equation (4:14) at N discrete points on the wire surface is equal to the choice of Dirac pulses as testing or weighting functions, the w_m , in equation (4:4). This choice of testing functions is often referred to as point matching and the advantage over other choices is that the integration associated with the inner product in equation (4:4) is avoided.

Finally, equation (4:15) can now be written in matrix form as :

$$[V_m] = [Z_{mn}] [\alpha_n] \quad (4:16)$$

Where the matrices are defined as :

$$[V_m] = \begin{bmatrix} E^i(y_1) \\ E^i(y_2) \\ \dots \\ E^i(y_N) \end{bmatrix} \quad [\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_N \end{bmatrix}$$

$$[Z_{mn}] = \frac{j\omega\mu}{4\pi} \begin{bmatrix} \int_0^{\Delta l} [1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2}] \frac{e^{-jkR_{11}}}{4\pi R_{11}} dy' & \int_0^{2\Delta l} [1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2}] \frac{e^{-jkR_{21}}}{4\pi R_{21}} dy' & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \int_0^{\Delta l} [1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2}] \frac{e^{-jkR_{12}}}{4\pi R_{12}} dy' & \int_0^{2\Delta l} [1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2}] \frac{e^{-jkR_{22}}}{4\pi R_{22}} dy' & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \int_0^{N\Delta l} [1 + \frac{1}{k^2} \frac{\partial^2}{\partial y'^2}] \frac{e^{-jkR_{NN}}}{4\pi R_{NN}} dy' \end{bmatrix}$$

The admittance matrix is the inverse of the impedance matrix and can by multiplication with the excitation matrix $[V_m]$ be used for calculations of the current distribution on the wire, $[\alpha_n]$. That is : $[\alpha_n] = [Y_{mn}] [V_m]$, where $[Y_{mn}]$ is the admittance matrix.

For transmitting wire antennas the excitation matrix represents the voltage excitation of the wire. If for instance the wire is excited by a voltage source at segment n, as shown in figure 4.3, the excitation matrix becomes :

$$[V_m] = \begin{bmatrix} 0 \\ 0 \\ \dots \\ V_n \\ \dots \\ 0 \\ 0 \end{bmatrix}$$

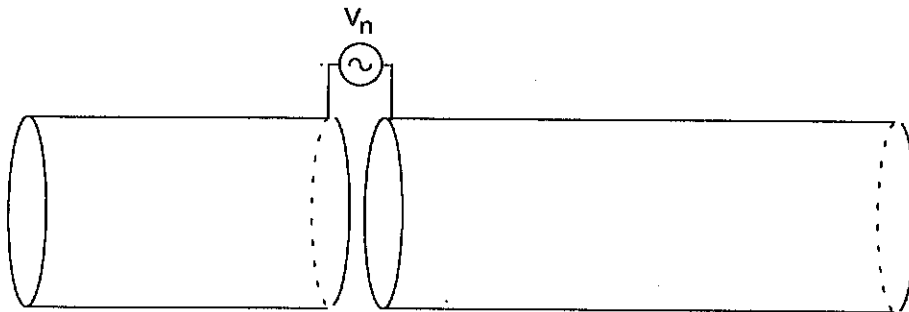


Figure 4.3. Wire antenna excited at segment n.

For wire scatters, figure 4:1 and 4:2, immersed in an electric field, E^i , the excitation matrix will be :

$$[V_m] = \begin{bmatrix} E^i(\frac{\Delta l}{2})\Delta l \cdot \hat{y} \\ E^i(\frac{3\Delta l}{2})\Delta l \cdot \hat{y} \\ \dots \\ E^i((N-\frac{1}{2})\Delta l)\Delta l \cdot \hat{y} \end{bmatrix}$$

where the y_m in equation (4:15) is chosen as the midpoint of each segment.

4.3 Computer codes

Many different codes based on the method of moments exist on the market. However, many of them are similar to a code originally developed at Lawrence Livermore National Laboratory which is called NEC (Numerical Electromagnetic Code).

NEC has the capability of analysing structures built-up with wires as well as surface patches. The code is based on two different integral equation formulations, EFIE (Electric Field Integral Equation) and MFIE (Magnetic Field Integral Equation). The EFIE is used for the wires and the MFIE is used for the patches. Even a lossy ground plane can be treated with a special module based on a Sommerfeld formulation. The code is not user-friendly in the sense that input data as well as output data are given in ASCII-files. However, there exist many input and output modules on the market. A simplified version of NEC called MININEC is sold by Artech House. This simplified version is developed for PCs and written in BASIC while the original code, NEC, is written in FORTRAN and intended for larger computers.

I-NAC-3 is a computer code made by Compact Software which is based on the method of moments. The code is using point matching and the basis functions used are hyperbolic functions. The structure that is about to be analysed has to be built up with wires. The structure is easily built with short commands and the result can directly be seen on the screen. Calculations are performed in the frequency domain but time domain results can be obtained by using the Fourier transform capability of the program. The code is designed to run on HP and SUN workstations under UNIX. The complexity of the problem which the program can handle depends on the available memory in the computer. As an example, 5000 nodes can be handled with a computer with 16 Mbytes of RAM.

Another simple code which can be run on PCs is the code "Analysis of Wire Antennas and Scatters" supplied by Artech house. The code is using point matching and the basis functions used are polynomials with a degree of up to nine. This code also can only handle structures built by wires. The code is designed for use on a personal computer compatible with DOS version 2.0 or later. The software is written in Fortran and C and the memory requirement is 512 kbytes RAM and a hard disk. Structures of a total length of about 30 wavelengths can be analysed.

5 Finite Element Method, FEM

The finite element method is a procedure for solving field problems by dividing the region of interest into small elements and by solving differential equations in each of those elements.

The method is usually employed in the frequency domain but can also be used for time domain problems.

5.1 General procedure

If FEM are about to be used for determination of the electromagnetic field in a region, the method can be summarised in the following steps :

- Divide the region under consideration into sub volumes
- Approximate the field in each of the sub volumes as a series of known functions with unknown coefficients
- Express the energy in each of the sub volumes as a function of the unknown coefficients
- Introduce constraints on the coefficients in order to satisfy boundary conditions
- Obtain the unique solution by minimising the energy

The procedure outlined above can be said to be based on a physical reasoning in the sense that the physical principle of minimising energy is used. However, the procedure is general and can be used for any functional equation, i.e. also for equations not associated with energies. Since it was stated in chapter 2.1 that the inhomogenous scalar Helmholtz equation often is of interest in electromagnetics, it is here used as an example of how the FEM matrices can be derived.

The scalar inhomogenous Helmholtz equation is :

$$\nabla^2 f + k^2 f = g \quad \text{in the volume } V \quad (5:1)$$

Where : f is the wanted scalar potential function
 g is a known driving (excitation) function

It can be shown [16] that the functional to minimise in order to solve the Helmholtz equation above is :

$$F = \frac{1}{2} \int_V [(\nabla f)^2 - k^2 f^2 + 2fg] dV \quad (5:2)$$

Thus, letting $\delta F=0$ gives the solution f .

In electromagnetics the first term in the integral (5:2) is associated with stored electric energy, the second term is associated with stored magnetic energy and losses (complex k) and the last term is associated with supplied energy.

The first step in FEM is to divide the region V into a number of sub regions, so-called elements. Let the functional for such an element be denoted as F_e .

$$F_e = \frac{1}{2} \int_e [(\nabla f)^2 - k^2 f^2 + 2fg] dV \quad (5:3)$$

The functional for the entire region, V , will then, by superposition, be :

$$F = \sum F_e$$

where the summation is to be taken over all elements.

The next step is to express the unknown function, f , for each element as a series of known functions with undetermined coefficients.

$$f = \sum_{i=1}^n \alpha_i f_i \quad (5:4)$$

Where : n is the number of nodes in the element

Equation (5:4) can also be expressed in matrix form as :

$$f = [\alpha]_e^T [f]_e = [\alpha_1 \quad \dots \quad \alpha_n]_e \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{bmatrix}_e \quad (5:5)$$

Where : T denotes a transpose and the index e is denoting the element

Insertion of equation (5:4) in (5:3) gives :

$$F_e = \frac{1}{2} \sum_i \sum_j \alpha_i (A_{ij} - k^2 B_{ij}) \alpha_j + \sum_i \alpha_i C_i$$

Where :

$$A_{ij} = \int_e \nabla f_i \nabla f_j dV$$

$$B_{ij} = \int_e f_i f_j dV$$

$$C_i = \int_e g f_i dV$$

or in matrix form as :

$$F_e = \frac{1}{2} [\alpha]_e^T \{ [A]_e - k^2 [B]_e \} [\alpha]_e + [\alpha]_e^T [C]_e \quad (5:6)$$

Summing up the expressions for all elements the functional for the entire region can be written as :

$$F = \frac{1}{2} [\alpha]^T \{ [A] - k^2 [B] \} [\alpha] + [\alpha]^T [C] \quad (5:7)$$

Where the matrices now are obtained as the sum of the components of all matrices for all elements.

The last step in the FEM procedure is to minimise the functional for the entire region. This can be done as :

$$\begin{aligned} \delta F = [0] &= \frac{\partial F}{\partial [\alpha]^T} \\ \Rightarrow \frac{1}{2} \{ [A] - k^2 [B] \} [\alpha] &= -[C] \end{aligned} \quad (5:8)$$

In equation (5:8) everything except the coefficients, α , is known. The unknown coefficients, α , are easily determined by matrix inversion.

$$[\alpha] = -2 \{ [A] - k^2 [B] \}^{-1} [C] \quad (5:9)$$

By this the solution of Helmholtz equation (5:1) is found, i.e. the f of equation (5:4).

5.2 Application to a lossy DC-transmission-line

In order to demonstrate the application of the finite element method a straight lossy two conductor DC-transmission-line is considered.

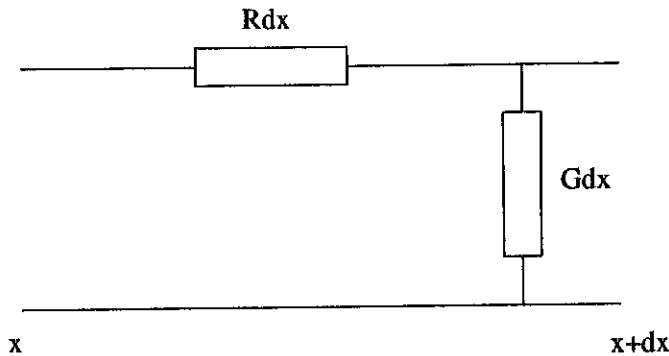


Figure 5.1. Infinitesimal section of a lossless two conductor transmission-line.

Referring to figure 5.1 the line parameters are the resistance, R and conductance, G , per unit length. Applying Kirchoff's voltage and current laws to the infinitesimal section, dx , gives :

$$\begin{aligned} \frac{dV}{dx} &= -RI(x) \\ \frac{dI}{dx} &= -GV(x) \end{aligned} \quad (5:10)$$

The task is to determine the line voltage as a function of the line co-ordinate, i.e. to solve equations (5:10) for $V(x)$.

The equations (5:10) can be transformed to Helmholtz equation by taking the derivative of the first equation with respect to x . By doing so and compare with equation (5:1) the following equalities are found : $f=V$, $k^2=-RG$ and $g=0$. Thus, the potential V can be found by equation (5:9).

Instead of taking this "mathematical" approach a more physical approach based on power minimisation is followed.

Divide the line into sub sections

The line is divided into N equally long sub sections as shown in figure 5.2. The choice of equally long sub sections is not necessary but is chosen here for the sake of simplicity .

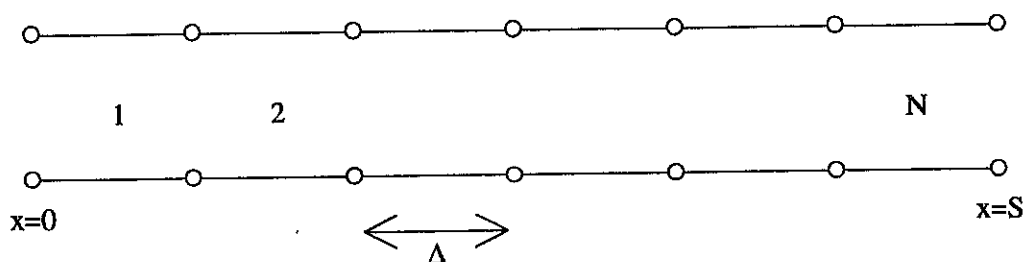


Figure 5.2. Line divided into N sections.

Referring to figure 5:2 : $\Delta = \frac{S}{N}$

Approximate the line voltage at each sub section

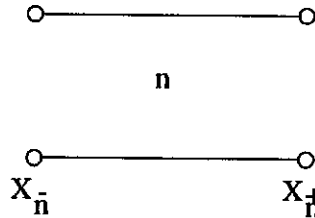


Figure 5:3. Sub section n.

From equations (5:10) it could be seen that the voltage should be differentiable, thus the voltage is approximated by a piece wise linear function.

$$V_n(x) = V_n^- \frac{x_+ - x}{x_+ - x_n^-} + V_n^+ \frac{x - x_n^-}{x_+ - x_n^-} = \frac{1}{\Delta} \left[V_n^- (x_+ - x) + V_n^+ (x - x_n^-) \right] \quad (5:11)$$

Express the power loss in each section

The power loss in an infinitesimal section, as shown in figure 5:1, can be obtained as the sum of the power loss in the resistance and the conductance, i.e. :

$$P_{dx} = \left[R dx |I(x)|^2 + G dx |V + dV|^2 \right] \approx \left[R dx |I(x)|^2 + G dx |V(x)|^2 \right] \quad (5:12)$$

Combining equations (5:10) and (5:12) and dividing with dx the power loss per unit length can be written in terms of the voltage as :

$$P = \left[R |I(x)|^2 + G |V(x)|^2 \right] = \left[\frac{1}{R} \left| \frac{dV}{dx} \right|^2 + G |V|^2 \right] \quad (5:13)$$

Thus, the total power loss in section n will be :

$$P_n = \int_{x_n^-}^{x_+} \left[\frac{1}{R} \left(\frac{dV_n}{dx} \right)^2 + G V_n^2 \right] dx \quad (5:14)$$

Evaluation of the integral using the expression for the voltage given by equation (5:11) results in :

$$P_n = \frac{1}{\Delta R} (V_n^+ - V_n^-)^2 + \Delta G \left[V_n^- V_n^+ + \frac{1}{3} (V_n^+ - V_n^-)^2 \right] \quad (5:15)$$

where : V_n^- and V_n^+ are voltages at the left and right end of section n respectively.

Introduce constraints on the coefficients

The voltage has to be continuous along the transmission-line. Thus, the voltage at the right end of section n has to be equal to the voltage at the left end of section n+1. The line also has to be excited somewhere. Taking this place to be the left end of the line, i.e. at $x=0$, the following relations must be valid :

$$V_1^- = V_{gen} \quad (\text{boundary condition at } x = 0)$$

$$V_n^+ = V_{n+1}^- ; n = 1..N-1$$

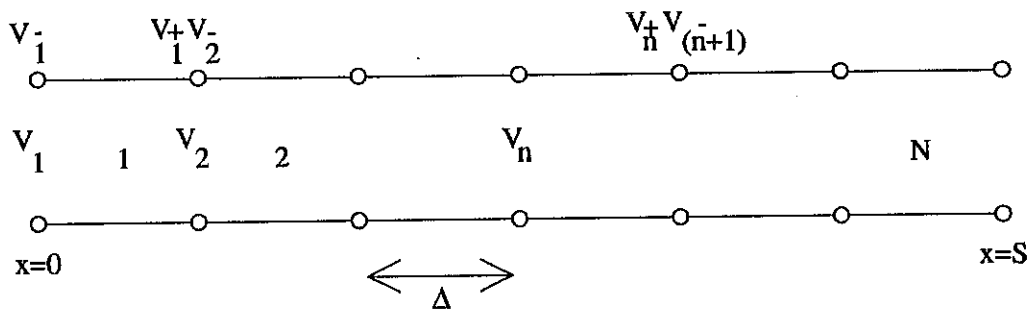


Figure 5:4. Numbering of the node voltages.

The constraints on the voltages imply that it is not necessary to keep apart the voltage at the right end of one section and the voltage at the left end of the subsequent section. Thus, as suggested in figure 5:4, the voltages are renumbered, i.e. V_n is denoting the voltage at node n (voltage at the left end of section n). With this renumbering, the power loss in section n, as described by equation (5:15), can be written as :

$$P_n = \frac{1}{\Delta R} (V_{n+1} - V_n)^2 + \Delta G \left[V_n V_{n+1} + \frac{1}{3} (V_{n+1} - V_n)^2 \right] ; n = 1..N \quad (5:16)$$

And the total power loss in the line is :

$$P_{tot} = \sum_{n=1}^N P_n \quad (5:17)$$

It should be noted that the number of sections is N and the number of node voltages is N+1. However, one of the voltages is fixed and known, i.e. the driving voltage $V_1 = V_{gen}$.

Minimise the overall power loss

The last step is to minimise the overall power loss with respect to the node voltages.

$$\text{Thus, letting } \frac{dP_{tot}}{dV_m} = 0 \quad ; \quad m = 2..N+1 \quad (5:18)$$

Note that the index m starts on 2. That is because the driving voltage $V_1=V_{gen}$ is not allowed to vary.

When solved on a computer, equation (5:18) is most conveniently written in matrix form. If the number of sections, N , is equal to six the matrices will be :

$$\begin{bmatrix} B & A & 0 & 0 & 0 & 0 \\ A & B & A & 0 & 0 & 0 \\ 0 & A & B & A & 0 & 0 \\ 0 & 0 & A & B & A & 0 \\ 0 & 0 & 0 & A & B & A \\ 0 & 0 & 0 & 0 & A & B/2 \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{bmatrix} = \begin{bmatrix} -V_1 A \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Where : } A = \frac{\Delta G}{3} - \frac{2}{\Delta R} \quad , \quad B = \frac{4\Delta G}{3} + \frac{4}{\Delta R}$$

This equation is easily solved for the unknown voltages, $V_2 - V_7$, by matrix inversion.

As an example the solution for the case of $R=1$ [mΩ/m], $G=1$ [mS/m] and $S=600$ [m] is shown in figure 5:5.

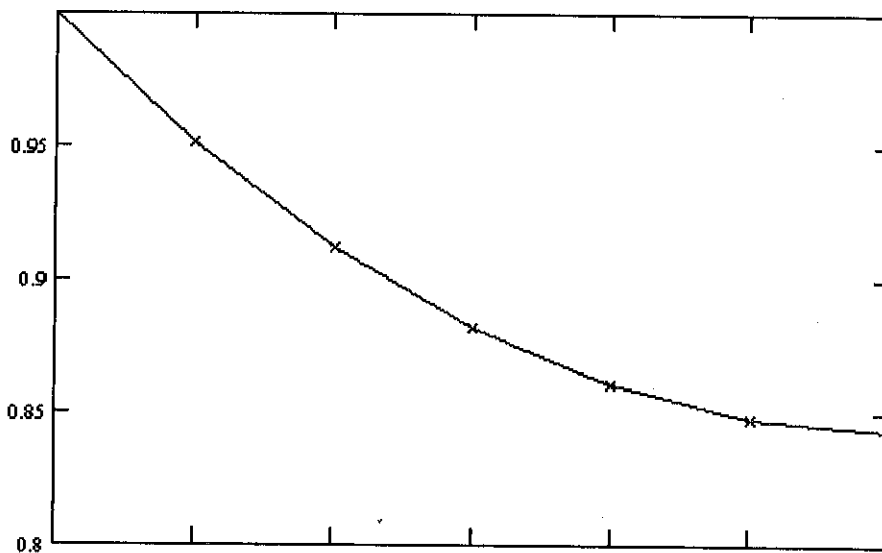


Figure 5:5. Line voltage as a function of the line co-ordinate. $S=600$; $R=1E-3$; $G=1E-3$.

5.3 Elements

Elements are the small regions in which the wanted function is approximated as a series of known functions with undetermined coefficients. Depending on the geometrical shape an element can be one-, two- or three-dimensional. Elements are also classified by what type of approximation function being used, so-called shape functions. Very often the shape function is a polynomial and the degree of the polynomial is called the order of the element.

Referring to the general procedure for the Helmholtz equation, outlined in chapter 5:1, the element is described by equation (5:4), i.e. :

$$f = \sum_{i=1}^n \alpha_i f_i$$

5.3.1 One-dimensional, first order

This is the simplest element and was the element used for the DC-transmission-line in chapter 5.2.

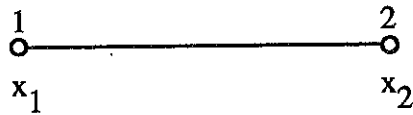


Figure 5:6. One-dimensional finite element of the first order.

For a first order element the function value along the element is assumed to vary linearly, thus : $V = a + bx$. As we shall see the element is fully described by the function values at the nodes and the co-ordinates of the nodes. Boundary conditions, of the Dirichlet type, can be fulfilled by forcing the function values at the nodes.

If the function values at $x=x_1$ and $x=x_2$ are V_1 and V_2 respectively (the node values) the following matrix relation must be valid :

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

solving for the coefficients a and b gives :

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

thus, the function $V = a + bx$ can be written as :

$$V = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

or :

$$V = \sum_{i=1}^2 V_i h_i(x)$$

where : $[h(x)] = [1 \quad x] \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1}$

When comparing with equation (5:5) in chapter 5.1, general procedure, it is found that :

$$[f]_e = [h(x)]^T \quad \text{and} \quad [\alpha]_e^T = [V]^T$$

5.3.2 Two-dimensional, first order

Many different shapes of two-dimensional finite elements are possible. Here is only the commonly used triangle shaped element considered. The triangle element is often used because it is easy to approximate an arbitrary surface with several triangle elements.

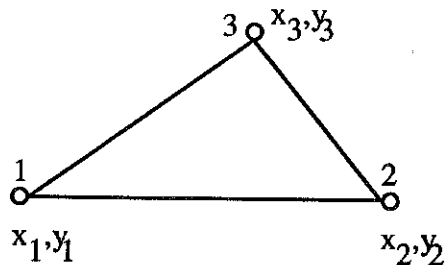


Figure 5:7. Two dimensional triangle finite element of the first order.

The function value for the first order triangle surface element is assumed to vary linearly in both co-ordinates, thus : $V = a + bx + cy$.

Using the same techniques as was used for the one dimensional element the following matrix equation for the triangle element can be obtained :

$$V = [1 \quad x \quad y] \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where V_1 , V_2 and V_3 are the function values at the nodes.

5.3.3 Three-dimensional, first order

The simplest three-dimensional element is the tetrahedron.

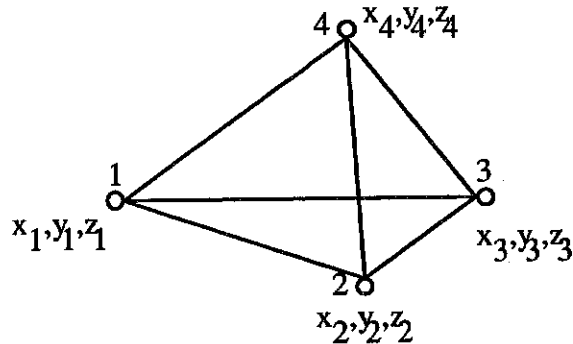


Figure 5:8. Three-dimensional tetrahedral finite element of the first order.

The function value for the first order tetrahedral volume element is assumed to vary linearly in all three co-ordinates, thus : $V = a + bx + cy + dz$.

In matrix form :

$$V = \begin{bmatrix} 1 & x & y & z \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

where V_1, V_2, V_3 and V_4 are the function values at the nodes.

5.4 Treatment of infinite regions

One of the disadvantages of the FEM is the fact that the entire region under consideration has to be modelled with elements. This means, for instance, that for an antenna problem the entire space has to be modelled with elements.

Since we cannot have an infinite number of elements, problems with boundaries at infinity have to be treated in a special way. Several techniques exist, such as : Artificial boundaries, Infinite elements, Combination of FEM and other methods.

The method of artificial boundaries is perhaps the simplest way to treat infinite regions. The technique is simply to replace the infinite region with a finite region bounded with an artificial boundary. The boundary can even be shifted in an iterative manner in order to satisfy a given criterion (such as the field strength) at the boundary .

The method of infinite elements can easily be implemented by letting one node, one side or one surface of the one-, two- or three-dimensional element respectively extend to infinity. The problem is, however, that the integral associated with the element (see Ch. 5.1) is diverging if the shape function of the element is not properly chosen. The choice

of linearly shape functions as was described in the preceding chapter will not do the job. Thus, the choice would be some type of a decaying function, preferably a function that describes the field behaviour fairly well.

The combination of FEM and other methods is performed in such a way that the FEM is used for the interior region, which is bounded, and some other method is used for the exterior region. The other method used for the exterior region could be an analytical method or a numerical method such as the method of moments. Finally the two solutions have to be combined, i.e. the fields at the common boundary have to be continuous.

5.5 Computer codes

There exist many FEM codes that can handle two-dimensional electromagnetic problems. For three-dimensional problems, on the other hand, the availability is poorer.

The code EMAS by Mac-Neal-Schwendler Corp. is an advanced code based on a three dimensional FEM formulation. The structure that is about to be analysed is fed into the computer in a CAD-like manner, i.e. the structure is drawn on the screen and the material properties can be defined by giving the material constants. When the user has defined the structure the computer divides the volume into finite elements. This division is performed in an intelligent way, meaning that the elements are smaller where the field variations are believed to be large. Special elements that makes it possible to analyse radiation problems are also available. When the computation is ready the output data can be presented in many different ways on the screen or on a plotter. Even post-calculations can be done in order to, for instance, determine power loss in a system. The code is designed to be run on workstations and more advanced computers.

6 Finite difference method, FDM

The finite difference method is a procedure for solving field problems by placing a regularly spaced grid of points over the region of interest and by solving differential equations at each point. The FDM can be applied in either the time domain or the frequency domain. The procedure was originally developed for time domain problems [17] and since time domain applications are the most common, this is the technique discussed in this chapter. The time domain procedure is often abbreviated as FDTD and the frequency domain procedure as FDFD.

6.1 General procedure of FDTD

The FDTD is a time and space stepping procedure. The first step in solving a field problem by FDTD is to divide the region under consideration into a number of small cells. The next step is to assume field values in all cells at the start time, often $t=0$. By approximating Maxwell's equations, eq. (2:1) and (2:2), by finite differences, the field values at all points and subsequent times can be evaluated by stepping through space and time. The time and space stepping is continued until a steady state is achieved or until the desired response is computed.

In the rectangular co-ordinate system Maxwell's equations, eq. (2:1) and (2:2), can be written as :

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = - \left(\frac{\partial B_x}{\partial t}, \frac{\partial B_y}{\partial t}, \frac{\partial B_z}{\partial t} \right) \quad (6:1)$$

$$\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = \left(\frac{\partial D_x}{\partial t}, \frac{\partial D_y}{\partial t}, \frac{\partial D_z}{\partial t} \right) + (J_x, J_y, J_z)$$

In order to discretize equations (6:1) the field components are assumed to be placed at different locations in a unit cell. The electric field components are placed at the midpoint of the corner lines of the cell and the magnetic field components are placed at the centre of the cell surfaces, see figure 6:1. The magnetic component placed at a certain surface is the component that represents the outward normal direction to the surface. The electric field component placed on a certain corner line is the component that represents the same direction as the co-ordinate axis parallel to the corner line.

If the corner point of the unit cell in figure 6:1 is denoted as (i,j,k) the Cartesian co-ordinates will be $(i\Delta x, j\Delta y, k\Delta z)$. If the time also is discretized as $t=n\Delta t$ a discretized version of an arbitrary function of space and time can be written as :

$$F(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = F^n(i, j, k) \quad (6:2)$$

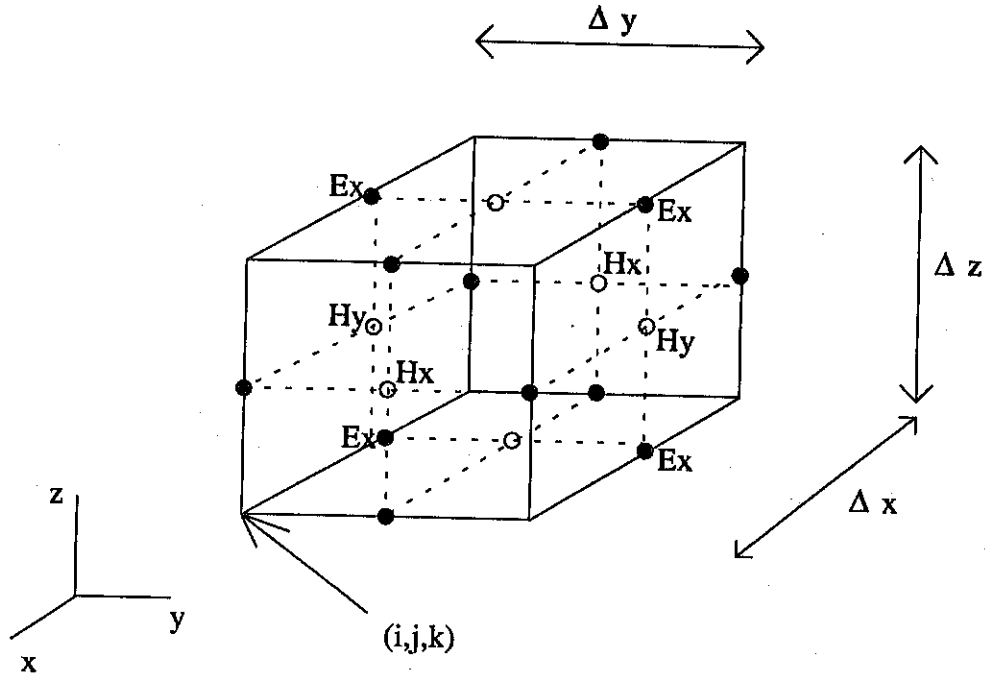


Figure 6:1. Unit cell.

Taking the x-component of the first equation in the set (6:1) as an example, thus :

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} = -\mu \frac{\partial H_x}{\partial t} \quad (6:3)$$

Using the notation introduced in equation (6:2) a first order finite difference approximation of (6:3) can be written as :

$$\begin{aligned} & \frac{E_z^n\left(i, j, k + \frac{1}{2}\right) - E_z^n\left(i, j + 1, k + \frac{1}{2}\right)}{\Delta y} - \frac{E_y^n\left(i, j + \frac{1}{2}, k + 1\right) - E_y^n\left(i, j + \frac{1}{2}, k\right)}{\Delta z} = \\ & = -\mu \frac{H_x^{n+\frac{1}{2}}\left(i, j + \frac{1}{2}, k + \frac{1}{2}\right) - H_x^{n-\frac{1}{2}}\left(i, j + \frac{1}{2}, k + \frac{1}{2}\right)}{\Delta t} \end{aligned} \quad (6:4)$$

Referring to figure 6:1 the field components in the equation are placed on the nearest surface.

And for the x -component of the second equation in the set (6:1) :

$$\frac{H_x^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) - H_x^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j-\frac{1}{2}, k\right)}{\Delta y} - \frac{H_y^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) - H_y^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j, k-\frac{1}{2}\right)}{\Delta z} = \epsilon \frac{E_x^n\left(i+\frac{1}{2}, j, k\right) - E_x^{n-1}\left(i+\frac{1}{2}, j, k\right)}{\Delta t} - J_x^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j, k\right) \quad (6:5)$$

The interpretation of equation (6:4) is that the magnetic field at time $(n+1/2)\Delta t$ can be determined by the knowledge of the electric field at time $n\Delta t$. And similarly for equation (6:5), the electric field at time $n\Delta t$ is determined by the knowledge of the magnetic field at time $(n-1/2)\Delta t$.

Using the same principle for the other equations in the set (6:1) gives a complete set of difference equations which makes it possible to step through the region under consideration in both space and time.

6.2 The unit cell and boundaries

The unit cell is shown in figure 6:1. Since a finite difference approximation of Maxwell's equations is constructed for every cell in the region, it is possible to have different material constants, ϵ and μ , for each of the cells. Thus, it is possible to analyse, for instance, wave propagation in an inhomogenous medium. It is also possible to have different material constants for different field orientations, this makes it possible to analyse anisotropic materials. The material constants can even be varied for different time steps according to a given criterion, such as the field strength, which makes it possible to analyse non-linear materials.

A very common boundary used in simulations is the perfect electric conductor. The boundary condition at such a boundary is that of vanishing tangential electric field, and consequently vanishing normal magnetic field. The unit cell described in the preceding chapter, figure 6:1, is particularly well suited for a perfect conducting boundary. This is so because when looking at the field components at one of the surfaces of the unit cell it is seen that the only field components on such a surface are tangential electric and normal magnetic fields.

6.3 Time and space step sizes

If field variations are to be seen in the computation the space step sizes, Δx , Δy and Δz , should be sufficiently small, i.e. small compared to the wavelength of the wanted response. In practise this means that the maximum space step size should not be greater than about one tenth of the wavelength.

For numerical stability it is necessary that the relation between space and time step sizes satisfies the following criterion :

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \geq c\Delta t = \sqrt{\frac{1}{\mu\epsilon}}\Delta t$$

If different μ and/or ϵ is used in the cells the maximum value of the speed of light, c , for a cell should be used in the formula.

6.4 Computer codes

The code EM Wavetracer by Wavetracer is a code based on a three-dimensional formulation of FDTD. Bounded as well as unbounded, for instance antenna, problems can be analysed. The excitation is defined as an incident plane wave with wave forms sinusoidal, gaussian or gaussian modulated sinusoidal. The code is specially developed for parallel computers but can also be run on ordinary computers.

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