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Abstract

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Time and frequency error models for atomic frequency standards are presented in this report, together with derivations of model parameters. The models are suited for use in Kalman filtering, e.g., for combining data from several frequency standards to form a “group clock”.

Key words: time metrology, frequency standards, Kalman filter

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1 Introduction

The purpose of this report is to document the equations in the Kalman filters used for clock data combinations ("group clocks"). The treatment of the stochastic processes is not mathematically rigorous; the derivations are intended as engineering tools for finding appropriate model parameters for the filtering.

2 Noise process models of clocks

We model the time offset, ϕ, of an atomic frequency standard on time intervals, say minutes to months, as the sum of three noise components

$$\phi = \phi_1 + \phi_2 + \phi_3 \tag{1}$$

where each term is originating from integrations of white noise sequences, $\sigma_i \, \nu_i$,

$$\frac{d\phi_1}{dt} = \sigma_1 \, \nu_1, \;\; \frac{d^2\phi_2}{dt^2} = \sigma_2 \, \nu_2, \;\; \frac{d^3\phi_3}{dt^3} = \sigma_3 \, \nu_3. \tag{2}$$

By definition, the sequences ν_i are normalized (see Appendix B), and scaled by the standard deviations, σ_i, in order to give appropriate size for the clock offsets ϕ_i.

Each term will have a time interval regime where its relative influence on the total time offset is at its largest. The first term, ϕ_1, will dominate on small enough time intervals, while ϕ_3 will dominate on the largest times intervals. On these intervals the definition of ϕ_3 from equation (2) will give rise to a parabolic time offset (see below). This is observed for, e.g., H-masers.

2.1 Noise sequence ν_3

Starting with the defining function

$$\frac{d^3\phi_3}{dt^3} = \sigma_3 \, \nu_3$$

and defining a frequency offset, f_3, and a frequency drift, a_3

$$f_3 \equiv \frac{d\phi_3}{dt}, \;\; a_3 \equiv \frac{df_3}{dt} = \frac{d^2\phi_3}{dt^2} \tag{3}$$

we get:

$$\frac{da_3}{dt} = \sigma_3 \, \nu_3 \tag{4}$$

Frequency drift, frequency offset and clock time offset at time t can now be expressed as once, twice, and thrice integration of the noise after a starting point t_0

$$\begin{aligned}
a_3(t) &= a_3(t_0) + \int_{t_0}^{t} \sigma_3 \, \nu_3(t^1) \, dt^1 \\
f_3(t) &= f_3(t_0) + (t - t_0) \, a_3(t_0) + \int_{t_0}^{t} (\int_{t_0}^{t^2} \sigma_3 \, \nu_3(t^1) \, dt^1) \, dt^2 \\
\phi_3(t) &= \phi_3(t_0) + (t - t_0) \, f_3(t_0) + \frac{(t - t_0)^2}{2} \, a_3(t_0) + \int_{t_0}^{t} (\int_{t_0}^{t^3} (\int_{t_0}^{t^2} \sigma_3 \, \nu_3(t^1) \, dt^1) \, dt^2) \, dt^3
\end{aligned} \tag{5}$$

Notice! Superscripts on t, e.g. t^3, are just indices; they do not denote exponents.

The data occur as samples at a set of discrete time instances, t_p. Inserting the present and previous sampling instants, t_p and t_{p-1}, as t and t_0, and defining $\tau_p \equiv t_p - t_{p-1}$ we get:

$$
\begin{aligned}
a_3(t_p) &= a_3(t_{p-1}) + \int_{t_{p-1}}^{t_p} \sigma_3\, \nu_3(t^1)\, dt^1 \\
f_3(t_p) &= f_3(t_{p-1}) + \tau_p\, a_3(t_{p-1}) + \int_{t_{p-1}}^{t_p} (\int_{t_{p-1}}^{t^2} \sigma_3\, \nu_3(t^1)\, dt^1)\, dt^2 \\
\phi_3(t_p) &= \phi_3(t_{p-1}) + \tau_p\, f_3(t_{p-1}) + \frac{\tau_p^2}{2}\, a_3(t_{p-1}) + \int_{t_{p-1}}^{t_p} (\int_{t_{p-1}}^{t^3} (\int_{t_{p-1}}^{t^2} \sigma_3\, \nu_3(t^1)\, dt^1)\, dt^2)\, dt^3
\end{aligned} \tag{6}
$$

Let us define the discrete function $w_{i,n}(p)$ as n times integration of the noise process $\sigma_i\, \nu_i$ in the interval between t_{p-1} and t_p. For n =1 to 3 we get:

$$
\begin{aligned}
w_{i,1}(p) &\equiv \int_{t_{p-1}}^{t_p} \sigma_i\, \nu_i(t^1)\, dt^1 \\
w_{i,2}(p) &\equiv \int_{t_{p-1}}^{t_p} (\int_{t_{p-1}}^{t^2} \sigma_i\, \nu_i(t^1)\, dt^1)\, dt^2 \\
w_{i,3}(p) &\equiv \int_{t_{p-1}}^{t_p} (\int_{t_{p-1}}^{t^3} (\int_{t_{p-1}}^{t^2} \sigma_i\, \nu_i(t^1)\, dt^1)\, dt^2)\, dt^3
\end{aligned} \tag{7}
$$

It is shown in Appendix A that the multidimensional integrals in equation (7) can be reduced to integrals in one dimension:

$$
\begin{aligned}
w_{i,n}(p) &\equiv \int_{t^n=t_{p-1}}^{t_p} (\int_{t^{n-1}=t_{p-1}}^{t^n} \cdots (\int_{t^1=t_{p-1}}^{t^2} \sigma_i\, \nu_i(t^1)\, dt^1) \cdots dt^{n-1})\, dt^n \\
&= \int_{t^1=t_{p-1}}^{t_p} \frac{(t_p - t^1)^{n-1}}{(n-1)!}\, \sigma_i\, \nu_i(t^1)\, dt^1
\end{aligned} \tag{8}
$$

Viewing $a_3(p), f_3(p), \phi_3(p)$ as discrete functions we can rewrite equation (6):

$$
\begin{aligned}
a_3(p) &= a_3(p-1) + w_{3,1}(p) \\
f_3(p) &= f_3(p-1) + \tau_p\, a_3(p-1) + w_{3,2}(p) \\
\phi_3(p) &= \phi_3(p-1) + \tau_p\, f_3(p-1) + \frac{\tau_p^2}{2}\, a_3(p-1) + w_{3,3}(p)
\end{aligned} \tag{9}
$$

In a stricter sense the functions in equation (9) are not identical with those with the same name in equation (5). They are merely composite functions of those in equation (5) with the function $t_s(p)$, e.g.,

$$a'_3(p) = a_3(t_s(p)) \tag{10}$$

where $t_s(p)$ defines the sampling time for each (integer) sample number p. However, in the following we have chosen to ignore the formal difference between the continuous and discrete versions of the functions, e.g., a'_3 and a_3.

2.2 Noise sequences ν_2 and ν_1

We start with the defining function for the second process

$$\frac{d^2\phi_2}{dt^2} = \sigma_2\,\nu_2$$

and define a frequency offset, f_2, as:

$$f_2 \equiv \frac{d\phi_2}{dt} \tag{11}$$

In analogy with the procedures for the third process above, the frequency offset and clock offset for the second process at two consecutive sampling instants can now be derived from once and twice integration of the noise process, followed by discretization:

$$\begin{aligned} f_2(p) &= f_2(p-1) + w_{2,1}(p) \\ \phi_2(p) &= \phi_2(p-1) + \tau_p\, f_2(p-1) + w_{2,2}(p) \end{aligned} \tag{12}$$

Finally, for the first process one integration of the defining function

$$\frac{d\phi_1}{dt} = \sigma_1\,\nu_1$$

give the relation between the clock offset at two consecutive sampling instants:

$$\phi_1(p) = \phi_1(p-1) + w_{1,1}(p) \tag{13}$$

3 State description of clocks

3.1 State equations

As a complement to the total clock offset, $\phi = \phi_1 + \phi_2 + \phi_3$, we can define (total) frequency offset and frequency drifts as:

$$\begin{aligned} f &\equiv f_2 + f_3 \\ a &\equiv a_3 \end{aligned} \tag{14}$$

Note that $f \neq d\phi/dt, a \neq df/dt$ for this combination of processes; only "smooth" components (i.e., involving integration of the white sequences) contribute to our definition of f and a. The parameters ϕ, f, and a constitute a state description of the clock error, that can be used to compile equations (9), (12), and (13) into the combination:

$$
\begin{aligned}
\phi(p) &= \phi(p-1) + \tau_p\, f(p-1) + \frac{\tau_p^2}{2}\, a(p-1) + w_{1,1}(p) + w_{2,2}(p) + w_{3,3}(p) \\
f(p) &= f(p-1) + \tau_p\, a(p-1) + w_{2,1}(p) + w_{3,2}(p) \\
a(p) &= a(p-1) + w_{3,1}(p)
\end{aligned} \tag{15}
$$

By defining a state vector, $\mathbf{x}$, and an accompanying noise vector $\mathbf{w}$, as

$$\mathbf{x} \equiv \begin{bmatrix}\phi & f & a\end{bmatrix}^T \tag{16}$$

$$\mathbf{w} \equiv \begin{bmatrix}w_{1,1} + w_{2,2} + w_{3,3} & w_{2,1} + w_{3,2} & w_{3,1}\end{bmatrix}^T \tag{17}$$

we can now rewrite equations (15) for a clock k as:

$$\mathbf{x}_k(p) = \Phi(p)\,\mathbf{x}_k(p-1) + \mathbf{w}_k(p) \tag{18}$$

where

$$\Phi(p) \equiv \begin{bmatrix}1 & \tau_p & \frac{\tau_p^2}{2} \\ 0 & 1 & \tau_p \\ 0 & 0 & 1\end{bmatrix} \tag{19}$$

3.2 Noise covariance

In Kalman filters where data from a set of clocks are compared, the covariance of the components in the process noises, $\mathbf{w}$, is essential for calculating the optimal filter parameters. The covariance for two clocks k and l can be written as:

$$Q_{kl}(p) \equiv E\{\mathbf{w}_k(p)\,\mathbf{w}_l^T(p)\} \tag{20}$$

where $E\{\}$ denote expectation value. This covariance matrix can be calculated using an expression for the covariance between noise processes w_{i,n_i} and w_{j,n_j}, integrated n_i and n_j times (see equations (7) and (8)):

$$E\big\{w_{i,n_i}\, w_{j,n_j}\big\} = \frac{c_{ij}\,\sigma_i\,\sigma_j\,\tau_p^{n_i+n_j-1}}{(n_i-1)!\,(n_j-1)!\,(n_i+n_j-1)} \tag{21}$$

where c_{ij} is the correlation coefficient between the noise sequences i and j. This relation is derived in Appendix B.

A general expressions for the elements in Q_{kl} of equation (20) is given in Appendix C. For the case of all six noise processes of clock k and l are uncorrelated we get:

$$
\begin{aligned}
Q_{kk}(p) &= \begin{bmatrix}
\sigma_{1k}^2\tau_p + \sigma_{2k}^2\frac{\tau_p^3}{3} + \sigma_{3k}^2\frac{\tau_p^5}{20} & \sigma_{2k}^2\frac{\tau_p^2}{2} + \sigma_{3k}^2\frac{\tau_p^4}{8} & \sigma_{3k}^2\frac{\tau_p^3}{6} \\
\sigma_{2k}^2\frac{\tau_p^2}{2} + \sigma_{3k}^2\frac{\tau_p^4}{8} & \sigma_{2k}^2\tau_p + \sigma_{3k}^2\frac{\tau_p^3}{3} & \sigma_{3k}^2\frac{\tau_p^2}{2} \\
\sigma_{3k}^2\frac{\tau_p^3}{6} & \sigma_{3k}^2\frac{\tau_p^2}{2} & \sigma_{3k}^2\tau_p
\end{bmatrix} \\
Q_{kl}(p) &= \mathbf{0},\, k \neq l
\end{aligned} \tag{22}
$$

4 State description of a first order Gauss-Markov process

In a first order Gauss-Markov process a negative feedback limits its amplitude. It could be used, e.g., as part of describing the measurement link between clocks. It is defined by the equation:

$$\frac{d\phi_g}{dt} = -\beta\,\phi_g + \sigma_g\,\nu_g \tag{23}$$

where the feedback factor β is positive for generating a negative feedback. Using $\psi(t) = e^{\beta t}\,\phi_g(t)$ this can be written

$$\frac{d\psi}{dt} = e^{\beta t}\sigma_g\,\nu_g$$

and after integration ant multiplication with $e^{-\beta t}$ we get

$$\phi_g(t) = e^{-\beta(t-t_0)}\,\phi_g(t_0) + \int_{t_0}^{t} e^{-\beta(t-t^1)}\sigma_g\,\nu_g(t^1)\,dt^1 \tag{24}$$

In analogy with the clock processes we discretize this process and get:

$$\phi_g(p) = e^{-\beta\tau_p}\,\phi_g(p-1) + w_g(p) \tag{25}$$

where

$$w_g(p) = \int_{t_{p-1}}^{t_p} e^{-\beta(t_p-t^1)}\sigma_g\,\nu_g(t^1)\,dt^1 \tag{26}$$

For Kalman filters using ϕ_g as a state variable the covariance between the noise w_g and other noise components is needed (in Q). In analogy with Appendix B we derive for two first order Gauss-Markov processes k and l:

$$\begin{aligned} E\{w_{gk}\,w_{gl}\} &= \sigma_{gk}\,\sigma_{gl} \int_{t^1=t_{p-1}}^{t_p} \int_{t^{1*}=t_{p-1}}^{t_p} e^{-\beta_k(t_p-t^1)}\,e^{-\beta_l(t_p-t^{1*})}\,E\{\nu_{gk}(t^1)\,\nu_{gl}(t^{1*})\}\,dt^1\,dt^{1*} \\ &= \sigma_{gk}\,\sigma_{gl} \int_{t^1=t_{p-1}}^{t_p} e^{-\beta_k(t_p-t^1)}\,e^{-\beta_l(t_p-t^1)}\,c_{gk,gl}\,dt^1 \\ &= \frac{c_{gk,gl}\,\sigma_{gk}\,\sigma_{gl}}{(\beta_k+\beta_l)}\left[1 - e^{-(\beta_k+\beta_l)\tau_p}\right] \end{aligned} \tag{27}$$

where $c_{gk,gl}$ is the correlation coefficient between the noise sequences ν_{gk} and ν_{gl}.

For the case the generating noise sequence in a Gauss-Markov process is correlated with the noise generating sequence of a clock process, w_{i,n_i}, we get:

$$
\begin{aligned}
&E\big\{w_{gk}\, w_{i,n_i}\big\} \\
&\quad = \frac{\sigma_{gk}\,\sigma_i}{(n_i-1)!} \int\limits_{t^1=t_{p-1}}^{t_p} \int\limits_{t^{1*}=t_{p-1}}^{t_p} e^{-\beta_k (t_p - t^1)}\,(t_p - t^{1*})^{n_i-1}\, E\{\nu_{gk}(t^1)\,\nu_i(t^{1*})\}\, dt^1\, dt^{1*} \\
&\quad = \frac{\sigma_{gk}\,\sigma_i}{(n_i-1)!} \int\limits_{t^1=t_{p-1}}^{t_p} e^{-\beta_k (t_p - t^1)}\,(t_p - t^1)^{n_i-1}\, c_{gk,i}\, dt^1 \\
&\quad = [\text{ repeated integration by parts }] \\
&\quad = \frac{c_{gk,i}\,\sigma_{gk}\,\sigma_i}{(n_i-1)!\,(-\beta_k)^{n_i}} \Big[e^{-\beta_k \tau_p} \Big[(-\beta\tau_p)^{n_i-1} - (n_i-1)(-\beta\tau_p)^{n_i-2} \\
&\qquad + (n_i-1)(n_i-2)(-\beta\tau_p)^{n_i-3} - \cdots + (-1)^{n_i-1}(n_i-1)! \Big] + (-1)^{n_i}(n_i-1)! \Big] \qquad (28)
\end{aligned}
$$

5 Triple difference variance

In order to find values for the parameters σ_i from measured clock offsets, $\phi(p)$, a chain of differentiations can be used. These can remove the dependence on the actual state of the clock (as defined by ϕ, f, and a) leaving a function of w terms from which conclusions on the standard deviations (σ) can be drawn.

We assume that the sampling points are equally spaced with separation τ, and define a differentation of a discrete function by $\Delta\psi(p) \equiv \psi(p) - \psi(p-1)$. By differentiating the measured phase offset three times, and estimating the variance of the differentiated data we get a polynomial in τ with coefficients based on the sought constants σ_i.

We start by differentiating the third phase offset ϕ_3. From equation (9) follows:

$$\Delta\phi_3(p) = \tau\, f_3(p-1) + \frac{\tau^2}{2} a_3(p-1) + w_{3,3}(p) \tag{29}$$

By using

$$\begin{aligned} &f_3(p-1) - f_3(p-2) = \tau\, a_3(p-2) + w_{3,2}(p-1) \text{ and} \\ &a_3(p-1) - a_3(p-2) = w_{3,1}(p-1) \end{aligned}$$

derived from equation (9) we can write an expression for the result after a second differentiation as:

$$\begin{aligned} \Delta\Delta\phi_3(p) \equiv &\Delta(\Delta\phi_3(p)) \\ =&\tau^2\, a_3(p-2) - \frac{\tau^2}{2} w_{3,1}(p-1) + \tau\, w_{3,2}(p-1) + w_{3,3}(p) - w_{3,3}(p-1) \end{aligned} \tag{30}$$

and finally, a third differentiation remove the remaining dependence on the state variables, and we get:

$$\begin{aligned} \Delta\Delta\Delta\phi_3(p) =& \frac{\tau^2}{2}[w_{3,1}(p-1) + w_{3,1}(p-2)] + \tau[w_{3,2}(p-1) - w_{3,2}(p-2)] \\ &+ w_{3,3}(p) - 2w_{3,3}(p-1) + w_{3,3}(p-2) \end{aligned} \tag{31}$$

Using equations (12) and (13) we can get the triple difference also for ϕ_2 and ϕ_1:

$$\Delta\Delta\Delta\phi_2(p) = \tau[w_{2,1}(p-1) - w_{2,1}(p-2)] + w_{2,2}(p) - 2w_{2,2}(p-1) + w_{2,2}(p-2) \tag{32}$$

$$\Delta\Delta\Delta\phi_1(p) = w_{1,1}(p) - 2w_{1,1}(p-1) + w_{1,1}(p-2) \tag{33}$$

We can now combine the three terms

$$T_1 = \Delta\Delta\Delta\phi_1(p),\, T_2 = \Delta\Delta\Delta\phi_2(p),\, T_3 = \Delta\Delta\Delta\phi_3(p)$$

to form the triple difference of ϕ and calculate its variance:

$$\begin{aligned} E\{(\Delta\Delta\Delta\phi)^2\} =&E\{(T_1 + T_2 + T_3)^2\} \\ =&E\{T_1^2\} + E\{T_2^2\} + E\{T_3^2\} + 2E\{T_1T_2\} + 2E\{T_1T_3\} + 2E\{T_2T_3\} \\ =&[\text{ See derivations in Appendix D }] \\ =&6\sigma_1^2\,\tau + \sigma_2^2\,\tau^3 - 2c_{13}\sigma_1\sigma_3\,\tau^3 + \frac{11}{20}\sigma_3^2\,\tau^5 \end{aligned} \tag{34}$$

Appendix A: Integration of the process noise

Below we derive the reduction of multidimensional integrals of the process noise into one integrals found in equation (8). Notice! Superscripts on t, e.g. t^n, are just indices (in the multidimensional space of integration); they do not denote exponents.

$$
\begin{aligned}
w_{i,n} &\equiv \int_{t^n=t_{p-1}}^{t_p} \Big(\int_{t^{n-1}=t_{p-1}}^{t^n} \cdots \Big(\int_{t^1=t_{p-1}}^{t^2} \sigma_i\, \nu_i(t^1)\, dt^1 \Big) \cdots dt^{n-1} \Big)\, dt^n \\
&= \big[\text{ using Heaviside function: } \theta(x) = 1 \text{ if } x \geqslant 0, = 0 \text{ otherwise } \big] \\
&= \int_{t^n=t_{p-1}}^{t_p} \int_{t^{n-1}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \theta(t^n - t^{n-1}) \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-1} dt^n \\
&= \big[\text{ change order of integration, start with } t^n \big] \\
&= \int_{t^{n-1}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[\int_{t^n=t_{p-1}}^{t_p} \theta(t^n - t^{n-1})\, dt^n \right] \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-1} \\
&= \int_{t^{n-1}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[\int_{t^n=t^{n-1}}^{t_p} 1 \cdot dt^n \right] \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-1} \\
&= \int_{t^{n-1}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[t_p - t^{n-1} \right] \theta(t^{n-1} - t^{n-2}) \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-1} \\
&= \int_{t^{n-2}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[\int_{t^{n-1}=t_{p-1}}^{t_p} (t_p - t^{n-1})\, \theta(t^{n-1} - t^{n-2})\, dt^{n-1} \right] \\
&\qquad\qquad \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-2} \\
&= \int_{t^{n-2}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[\int_{t^{n-1}=t^{n-2}}^{t_p} (t_p - t^{n-1})\, dt^{n-1} \right] \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-2} \\
&= \int_{t^{n-2}=t_{p-1}}^{t_p} \cdots \int_{t^1=t_{p-1}}^{t_p} \left[\frac{(t_p - t^{n-2})^2}{2} \right] \theta(t^{n-2} - t^{n-3}) \cdots \theta(t^2 - t^1)\, \sigma_i\, \nu_i(t^1)\, dt^1 \cdots dt^{n-2} \\
&= \big[\text{ after a total of } n-1 \text{ integrations: } \big] \\
&= \int_{t^1=t_{p-1}}^{t_p} \frac{(t_p - t^1)^{n-1}}{(n-1)!}\, \sigma_i\, \nu_i(t^1)\, dt^1
\end{aligned}
$$

Appendix B: Derivation of process noise covariances

The sequences denoted ν in this report are white and normalized. This means that

$$\int_D f(t^1) E\{\nu_i(t^1)\,\nu_j(t)\}\,dt^1 = f(t)\,c_{ij},\ t \in D$$
$$= 0,\ t \notin D$$

where f is (almost) any function, and c_{ij} is the correlation coefficient between the two sequences. Especially the normalization give

$$\int_D E\{\nu_i(t^1)\,\nu_i(t)\}\,dt^1 = 1,\ t \in D$$

We also assume that the distribution functions of the stochastic processes $\pi(t)$ are such that the order between forming an expectation value and integration in time are interchangeable, i.e.,

$$E\Big\{\int \pi(t)\,dt\Big\} = \int E\Big\{\pi(t)\Big\}\,dt$$

With this background we can calculate the covariance between two noise processes $w_{i,n_i}(p)$ and $w_{j,n_j}(p)$ derived from integration of white noise sequences n_i and n_j times as defined in equation (8).

$$
\begin{aligned}
&E\{w_{i,n_i}(p)\,w_{j,n_j}(p)\}\\
&\quad = E\Bigg\{\int_{t^1=t_{p-1}}^{t_p} \frac{(t_p-t^1)^{n_i-1}}{(n_i-1)!}\,\sigma_i\,\nu_i(t^1)\,dt^1 \cdot \int_{t^{1*}=t_{p-1}}^{t_p} \frac{(t_p-t^{1*})^{n_j-1}}{(n_j-1)!}\,\sigma_j\,\nu_j(t^{1*})\,dt^{1*}\Bigg\}\\
&\quad = \frac{\sigma_i\,\sigma_j}{(n_i-1)!\,(n_j-1)!}\int_{t^1=t_{p-1}}^{t_p}\int_{t^{1*}=t_{p-1}}^{t_p} (t_p-t^1)^{n_i-1}\,(t_p-t^{1*})^{n_j-1}\,E\{\nu_i(t^1)\,\nu_j(t^{1*})\}\,dt^1\,dt^{1*}\\
&\quad = \frac{\sigma_i\,\sigma_j}{(n_i-1)!\,(n_j-1)!}\int_{t^1=t_{p-1}}^{t_p} (t_p-t^1)^{n_i-1}\,(t_p-t^1)^{n_j-1}\,c_{ij}\,dt^1\\
&\quad = \frac{c_{ij}\,\sigma_i\,\sigma_j}{(n_i-1)!\,(n_j-1)!}\int_{t^1=t_{p-1}}^{t_p} (t_p-t^1)^{n_i+n_j-2}\,dt^1\\
&\quad = \frac{c_{ij}\,\sigma_i\,\sigma_j\,(t_p-t_{p-1})^{n_i+n_j-1}}{(n_i-1)!\,(n_j-1)!\,(n_i+n_j-1)}\\
&\quad = \frac{c_{ij}\,\sigma_i\,\sigma_j\,\tau_p^{n_i+n_j-1}}{(n_i-1)!\,(n_j-1)!\,(n_i+n_j-1)}
\end{aligned}
$$

Appendix C: Full clock state noise covariance matrix

The noise in clock offset, frequency offset, and frequency drift is:

$$\mathbf{w} = \begin{bmatrix} w_{1,1} + w_{2,2} + w_{3,3} & w_{2,1} + w_{3,2} & w_{3,1} \end{bmatrix}^T$$

For clocks k and l the noise covariance can be written as:

$$Q_{kl}(p) \equiv E\{\mathbf{w}_k(p)\,\mathbf{w}_l^T(p)\} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

where, e.g., $q_{12} = E\big\{(w_{1k,1} + w_{2k,2} + w_{3k,3}) \cdot (w_{2l,1} + w_{3l,2})\big\}$. Using the general expression for process noise covariance (derived in Appendix B)

$$E\big\{w_{i,n_i}\, w_{j,n_j}\big\} = \frac{c_{ij}\,\sigma_i\,\sigma_j\,\tau_p^{n_i+n_j-1}}{(n_i-1)!\,(n_j-1)!\,(n_i+n_j-1)}$$

we can express each element in Q based on the process standard deviations σ, the correlation c between the processes, and the time step τ_p as follows:

$$\begin{aligned}
q_{11} &= c_{1k,1l}\sigma_{1k}\sigma_{1l}\tau_p + c_{1k,2l}\sigma_{1k}\sigma_{2l}\frac{\tau_p^2}{2} + c_{1k,3l}\sigma_{1k}\sigma_{3l}\frac{\tau_p^3}{6} \\
&\quad + c_{2k,1l}\sigma_{2k}\sigma_{1l}\frac{\tau_p^2}{2} + c_{2k,2l}\sigma_{2k}\sigma_{2l}\frac{\tau_p^3}{3} + c_{2k,3l}\sigma_{2k}\sigma_{3l}\frac{\tau_p^4}{8} \\
&\quad + c_{3k,1l}\sigma_{3k}\sigma_{1l}\frac{\tau_p^3}{6} + c_{3k,2l}\sigma_{3k}\sigma_{2l}\frac{\tau_p^4}{8} + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^5}{20} \\
q_{12} &= c_{1k,2l}\sigma_{1k}\sigma_{2l}\tau_p + c_{1k,3l}\sigma_{1k}\sigma_{3l}\frac{\tau_p^2}{2} + c_{2k,2l}\sigma_{2k}\sigma_{2l}\frac{\tau_p^2}{2} \\
&\quad + c_{2k,3l}\sigma_{2k}\sigma_{3l}\frac{\tau_p^3}{3} + c_{3k,2l}\sigma_{3k}\sigma_{2l}\frac{\tau_p^3}{6} + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^4}{8} \\
q_{13} &= c_{1k,3l}\sigma_{1k}\sigma_{3l}\tau_p + c_{2k,3l}\sigma_{2k}\sigma_{3l}\frac{\tau_p^2}{2} + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^3}{6} \\
q_{21} &= c_{2k,1l}\sigma_{2k}\sigma_{1l}\tau_p + c_{2k,2l}\sigma_{2k}\sigma_{2l}\frac{\tau_p^2}{2} + c_{2k,3l}\sigma_{2k}\sigma_{3l}\frac{\tau_p^3}{6} \\
&\quad + c_{3k,1l}\sigma_{3k}\sigma_{1l}\frac{\tau_p^2}{2} + c_{3k,2l}\sigma_{3k}\sigma_{2l}\frac{\tau_p^3}{3} + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^4}{8} \\
q_{22} &= c_{2k,2l}\sigma_{2k}\sigma_{2l}\tau_p + c_{2k,3l}\sigma_{2k}\sigma_{3l}\frac{\tau_p^2}{2} + c_{3k,2l}\sigma_{3k}\sigma_{2l}\frac{\tau_p^2}{2} \\
&\quad + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^3}{3} \\
q_{23} &= c_{2k,3l}\sigma_{2k}\sigma_{3l}\tau_p + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^2}{2} \\
q_{31} &= c_{3k,1l}\sigma_{3k}\sigma_{1l}\tau_p + c_{3k,2l}\sigma_{3k}\sigma_{2l}\frac{\tau_p^2}{2} + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^3}{6} \\
q_{32} &= c_{3k,2l}\sigma_{3k}\sigma_{2l}\tau_p + c_{3k,3l}\sigma_{3k}\sigma_{3l}\frac{\tau_p^2}{2} \\
q_{33} &= c_{3k,3l}\sigma_{3k}\sigma_{3l}\tau_p
\end{aligned}$$

Appendix D: Derivation of triple difference variance

With $\phi = \phi_1 + \phi_2 + \phi_3$ and each triple difference term denoted

$$T_1 = \Delta\Delta\Delta\phi_1(p),\, T_2 = \Delta\Delta\Delta\phi_2(p),\, T_3 = \Delta\Delta\Delta\phi_3(p)$$

we get the variance of the triple difference of ϕ as:

$$\begin{aligned} E\{(\Delta\Delta\Delta\phi)^2\} =& E\{(T_1 + T_2 + T_3)^2\} \\ =& E\{T_1^2\} + E\{T_2^2\} + E\{T_3^2\} + 2E\{T_1T_2\} + 2E\{T_1T_3\} + 2E\{T_2T_3\} \end{aligned}$$

Below we derive an expression for each term using the process noice covariance equation

$$E\big\{w_{i,n_i}\, w_{j,n_j}\big\} = \frac{c_{ij}\,\sigma_i\,\sigma_j\,\tau_p^{n_i+n_j-1}}{(n_i-1)!\,(n_j-1)!\,(n_i+n_j-1)}$$

derived in Appendix B. We have also used the fact that the processes are white,

$$E\big\{w_{i,n_i}(p)\, w_{j,n_j}(q)\big\} = 0, p \neq q$$

i.e., we get covariance contributions only from identical time intervals.

$$\begin{aligned} &E\{T_1^2\} \\ &\quad= E\big\{\big[w_{1,1}(p) - 2w_{1,1}(p-1) + w_{1,1}(p-2)\big]^2\big\} \\ &\quad= [\text{ Covariance contributions only from identical time intervals }] \\ &\quad= E\big\{w_{1,1}^2(p)\big\} + 4E\big\{w_{1,1}^2(p-1)\big\} + E\big\{w_{1,1}^2(p-2)\big\} \\ &\quad= [\text{ Statistics independent of which time interval used }] \\ &\quad= 6E\{w_{1,1}^2\} \\ &\quad= 6\sigma_1^2\tau \end{aligned}$$

$$
\begin{aligned}
&E\{T_2^2\}\\
&\quad = E\Big\{\big[\tau[w_{2,1}(p-1) - w_{2,1}(p-2)] + w_{2,2}(p) - 2w_{2,2}(p-1) + w_{2,2}(p-2)\big]^2\Big\}\\
&\quad = [\text{ Covariance contributions only from identical time intervals }]\\
&\quad = E\big\{w_{2,2}^2(p)\big\}\\
&\qquad + E\Big\{\big[\tau w_{2,1}(p-1) - 2w_{2,2}(p-1)\big]^2\Big\}\\
&\qquad + E\Big\{\big[-\tau w_{2,1}(p-2) + w_{2,2}(p-2)\big]^2\Big\}\\
&\quad = [\text{ Statistics independent of which time interval used }]\\
&\quad = E\{w_{2,2}^2\}\\
&\qquad + \tau^2 E\{w_{2,1}^2\} + 4E\{w_{2,2}^2\} - 4\tau E\{w_{2,1}\, w_{2,2}\}\\
&\qquad + \tau^2 E\{w_{2,1}^2\} + E\{w_{2,2}^2\} - 2\tau E\{w_{2,1}\, w_{2,2}\}\\
&\quad = 2\tau^2 E\{w_{2,1}^2\} + 6E\{w_{2,2}^2\} - 6\tau E\{w_{2,1}\, w_{2,2}\}\\
&\quad = \sigma_2^2\big[2\tau^2 \cdot \tau + 6 \cdot \frac{\tau^3}{3} - 6\tau \cdot \frac{\tau^2}{2}\big]\\
&\quad = \sigma_2^2 \tau^3
\end{aligned}
$$

$$
\begin{aligned}
&E\{T_3^2\}\\
&\quad =E\big\{\big[\frac{\tau^2}{2}[w_{3,1}(p-1)+w_{3,1}(p-2)]+\tau[w_{3,2}(p-1)-w_{3,2}(p-2)]\\
&\qquad +w_{3,3}(p)-2w_{3,3}(p-1)+w_{3,3}(p-2)\big]^2\big\}\\
&\quad =\text{[Covariance contributions only from identical time intervals]}\\
&\quad =E\big\{w_{3,3}^2(p)\big\}\\
&\qquad +E\big\{\big[\frac{\tau^2}{2}w_{3,1}(p-1)+\tau w_{3,2}(p-1)-2w_{3,3}(p-1)\big]^2\big\}\\
&\qquad +E\big\{\big[\frac{\tau^2}{2}w_{3,1}(p-2)-\tau w_{3,2}(p-2)+w_{3,3}(p-2)\big]^2\big\}\\
&\quad =\text{[Statistics independent of which time interval used]}\\
&\quad =E\{w_{3,3}^2\}\\
&\qquad +\frac{\tau^4}{4}E\{w_{3,1}^2\}+\tau^2E\{w_{3,2}^2\}+4E\{w_{3,3}^2\}\\
&\qquad +\tau^3E\{w_{3,1}\,w_{3,2}\}-2\tau^2E\{w_{3,1}\,w_{3,3}\}-4\tau E\{w_{3,2}\,w_{3,3}\}\\
&\qquad +\frac{\tau^4}{4}E\{w_{3,1}^2\}+\tau^2E\{w_{3,2}^2\}+E\{w_{3,3}^2\}\\
&\qquad -\tau^3E\{w_{3,1}\,w_{3,2}\}+\tau^2E\{w_{3,1}\,w_{3,3}\}-2\tau E\{w_{3,2}\,w_{3,3}\}\\
&\quad =\frac{\tau^4}{2}E\{w_{3,1}^2\}+2\tau^2E\{w_{3,2}^2\}+6E\{w_{3,3}^2\}-\tau^2E\{w_{3,1}\,w_{3,3}\}-6\tau E\{w_{3,2}\,w_{3,3}\}\\
&\quad =\sigma_3^2\big[\frac{\tau^4}{2}\cdot\tau+2\tau^2\cdot\frac{\tau^3}{3}+6\cdot\frac{\tau^5}{20}-\tau^2\cdot\frac{\tau^3}{6}-6\tau\cdot\frac{\tau^4}{8}\big]\\
&\quad =\frac{11}{20}\sigma_3^2\tau^5
\end{aligned}
$$

$$
\begin{aligned}
&E\{T_1T_2\}\\
&\quad = E\big\{\big[w_{1,1}(p) - 2w_{1,1}(p-1) + w_{1,1}(p-2)\big]\cdot\\
&\qquad \big[\tau[w_{2,1}(p-1) - w_{2,1}(p-2)] + w_{2,2}(p) - 2w_{2,2}(p-1) + w_{2,2}(p-2)\big]\big\}\\
&\quad = [\text{ Covariance contributions only from identical time intervals }]\\
&\quad = E\big\{w_{1,1}(p)\cdot w_{2,2}(p)\big\}\\
&\qquad + E\big\{-2w_{1,1}(p-1)\cdot\big[\tau w_{2,1}(p-1) - 2w_{2,2}(p-1)\big]\big\}\\
&\qquad + E\big\{w_{1,1}(p-2)\cdot\big[-\tau w_{2,1}(p-2) + w_{2,2}(p-2)\big]\big\}\\
&\quad = [\text{ Statistics independent of which time interval used }]\\
&\quad = E\{w_{1,1}\,w_{2,2}\}\\
&\qquad - 2\tau E\{w_{1,1}\,w_{2,1}\} + 4E\{w_{1,1}\,w_{2,2}\}\\
&\qquad - \tau E\{w_{1,1}\,w_{2,1}\} + E\{w_{1,1}\,w_{2,2}\}\\
&\quad = -3\tau E\{w_{1,1}\,w_{2,1}\} + 6E\{w_{1,1}\,w_{2,2}\}\\
&\quad = c_{12}\sigma_1\sigma_2\big[-3\tau\cdot\tau + 6\cdot\frac{\tau^2}{2}\big]\\
&\quad = 0
\end{aligned}
$$

$$
\begin{aligned}
&E\{T_1T_3\}\\
&\quad = E\big\{\big[w_{1,1}(p) - 2w_{1,1}(p-1) + w_{1,1}(p-2)\big]\cdot\\
&\qquad \big[\frac{\tau^2}{2}[w_{3,1}(p-1) + w_{3,1}(p-2)] + \tau[w_{3,2}(p-1) - w_{3,2}(p-2)]\\
&\qquad + w_{3,3}(p) - 2w_{3,3}(p-1) + w_{3,3}(p-2)\big]\big\}\\
&\quad = [\text{ Covariance contributions only from identical time intervals }]\\
&\quad = E\big\{w_{1,1}(p)\cdot w_{3,3}(p)\big\}\\
&\qquad + E\big\{-2w_{1,1}(p-1)\cdot\big[\frac{\tau^2}{2}w_{3,1}(p-1) + \tau w_{3,2}(p-1) - 2w_{3,3}(p-1)\big]\big\}\\
&\qquad + E\big\{w_{1,1}(p-2)\cdot\big[\frac{\tau^2}{2}w_{3,1}(p-2) - \tau w_{3,2}(p-2) + w_{3,3}(p-2)\big]\big\}\\
&\quad = [\text{ Statistics independent of which time interval used }]\\
&\quad = E\{w_{1,1}\,w_{3,3}\}\\
&\qquad - \tau^2 E\{w_{1,1}\,w_{3,1}\} - 2\tau E\{w_{1,1}\,w_{3,2}\} + 4E\{w_{1,1}\,w_{3,3}\}\\
&\qquad + \frac{\tau^2}{2}E\{w_{1,1}\,w_{3,1}\} - \tau E\{w_{1,1}\,w_{3,2}\} + E\{w_{1,1}\,w_{3,3}\}\\
&\quad = -\frac{\tau^2}{2}E\{w_{1,1}\,w_{3,1}\} - 3\tau E\{w_{1,1}\,w_{3,2}\} + 6E\{w_{1,1}\,w_{3,3}\}\\
&\quad = c_{13}\sigma_1\sigma_3\big[-\frac{\tau^2}{2}\cdot\tau - 3\tau\cdot\frac{\tau^2}{2} + 6\cdot\frac{\tau^3}{6}\big]\\
&\quad = -c_{13}\sigma_1\sigma_3\tau^3
\end{aligned}
$$

$$
\begin{aligned}
&E\{T_2T_3\} \\
&\quad = E\Big\{\big[\tau[w_{2,1}(p-1) - w_{2,1}(p-2)] + w_{2,2}(p) - 2w_{2,2}(p-1) + w_{2,2}(p-2)\big]\cdot \\
&\quad\quad \big[\frac{\tau^2}{2}[w_{3,1}(p-1) + w_{3,1}(p-2)] + \tau[w_{3,2}(p-1) - w_{3,2}(p-2)] \\
&\quad\quad + w_{3,3}(p) - 2w_{3,3}(p-1) + w_{3,3}(p-2)\big]\Big\} \\
&\quad = [\text{ Covariance contributions only from identical time intervals }] \\
&\quad = E\big\{w_{2,2}(p)\cdot w_{3,3}(p)\big\} \\
&\quad\quad + E\big\{\big[\tau w_{2,1}(p-1) - 2w_{2,2}(p-1)\big]\cdot\big[\frac{\tau^2}{2}w_{3,1}(p-1) + \tau w_{3,2}(p-1) - 2w_{3,3}(p-1)\big]\big\} \\
&\quad\quad + E\big\{\big[-\tau w_{2,1}(p-2) + w_{2,2}(p-2)\big]\cdot\big[\frac{\tau^2}{2}w_{3,1}(p-2) - \tau w_{3,2}(p-2) + w_{3,3}(p-2)\big]\big\} \\
&\quad = [\text{ Statistics independent of which time interval used }] \\
&\quad = E\{w_{2,2}\, w_{3,3}\} \\
&\quad\quad + \frac{\tau^3}{2}E\{w_{2,1}\, w_{3,1}\} + \tau^2 E\{w_{2,1}\, w_{3,2}\} - 2\tau E\{w_{2,1}\, w_{3,3}\} \\
&\quad\quad - \tau^2 E\{w_{2,2}\, w_{3,1}\} - 2\tau E\{w_{2,2}\, w_{3,2}\} + 4E\{w_{2,2}\, w_{3,3}\} \\
&\quad\quad - \frac{\tau^3}{2}E\{w_{2,1}\, w_{3,1}\} + \tau^2 E\{w_{2,1}\, w_{3,2}\} - \tau E\{w_{2,1}\, w_{3,3}\} \\
&\quad\quad + \frac{\tau^2}{2}E\{w_{2,2}\, w_{3,1}\} - \tau E\{w_{2,2}\, w_{3,2}\} + E\{w_{2,2}\, w_{3,3}\} \\
&\quad = 2\tau^2 E\{w_{2,1}\, w_{3,2}\} - 3\tau E\{w_{2,1}\, w_{3,3}\} - \frac{\tau^2}{2}E\{w_{2,2}\, w_{3,1}\} \\
&\quad\quad - 3\tau E\{w_{2,2}\, w_{3,2}\} + 6E\{w_{2,2}\, w_{3,3}\} \\
&\quad = c_{23}\sigma_2\sigma_3\big[2\tau^2\cdot\frac{\tau^2}{2} - 3\tau\cdot\frac{\tau^3}{6} - \frac{\tau^2}{2}\cdot\frac{\tau^2}{2} - 3\tau\cdot\frac{\tau^3}{3} + 6\cdot\frac{\tau^4}{8}\big] \\
&\quad = 0
\end{aligned}
$$

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