Objective Functions for Balance in Traffic Engineering

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Abstract. We prove a result concerning objective functions that can be used to obtain efficient and balanced solutions to the multi-commodity network flow problem. This type of solution is of interest when routing traffic in the Internet. A particular case of the result proved here (see Corollary 2 below) was stated without proof in a previous paper.
Objective Functions for Balance in Traffic Engineering*

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1 Introduction

Traffic engineering encompasses performance evaluation and performance optimisation of operational IP networks. An important goal with traffic engineering is to use the available network resources more efficiently for different types of load patterns and to avoid congestion by having a relatively balanced distribution of traffic over the network.

Current routing protocols in the Internet calculate the shortest path to a destination in some metric without knowing anything about the traffic demand. Manual configuration by the network operator is therefore necessary to balance load between available paths to avoid congestion. One way of simplifying the task of the operator and improve use of the available network resources is to make the routing protocol sensitive to traffic demand. Routing then becomes a flow optimisation problem.

In another paper [1] we discussed a new routing algorithm based on multi-commodity flow optimisation. In this report we present and prove a theorem that has, as a special case, a result stated without proof in the mentioned paper. The theorem concerns optimisation objective functions which allow the network operator to choose a maximum desired link utilisation level. The optimisation will then find the most efficient solution, if it exists, satisfying this constraint. The objective function thus enables the operator to control the trade-off between minimising the network utilisation and balancing load over multiple paths.

We have tried to make the paper self-contained by giving enough background information. The interested reader is referred to a previous paper [1] and the references quoted there for more information on IP networks and on optimisation.

The rest of this report is organised as follows. Section 2 recalls the formulation of the multi-commodity flow problem as given in an earlier paper [1]. Section 3

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defines efficiency, balance and the objective functions of interest. The theorem and its proof are presented in Section 4.

2 Formulation of the multi-commodity flow problem

The routing problem in a network consists in finding a path or multiple paths that send the requested traffic through the network without exceeding the capacity of the links. In a previous paper [1] we modelled the routing problem as a multi-commodity network flow problem (MCF) as follows.

We represent the network by a directed graph $G = (N, E)$, where $N$ is a set of nodes and $E$ is a set of (directed) edges. We assume the graph is such that its edges can be uniquely represented by an ordered pair $(i, j)$ of nodes, where $i$ is the initial point of the edge and $j$ its final point. Every edge $(i, j) \in E$ has an associated capacity $k_{ij}$ reflecting the bandwidth available to the corresponding link. In addition, we assume given a demand matrix $D = D(s, t)$ expressing the traffic demand from node $s$ to node $t$ in the network. The entries of the demand matrix are non-negative and, to avoid trivialities, we assume that $D(s, t) > 0$ for at least one pair of nodes. We model commodities as (only destination) nodes, i.e., a commodity $t$ is to be interpreted as “all traffic to $t$”. Then the corresponding (MCF) problem can be formulated as follows:

$$\min \{ f(y) \mid y \in P_{12} \}$$

(MCF$_{12}$)

where $y = (y_{ij}^t)$, for $t \in N$, $(i, j) \in E$, and $P_{12}$ is the polyhedron defined by the equations:

$$\sum_{\{j \mid (i, j) \in E\}} y_{ij}^t - \sum_{\{j \mid (j, i) \in E\}} y_{ji}^t = d(i, t) \quad \forall i, t \in N \quad (1)$$

$$\sum_{i \in N} y_{ij}^t \leq k_{ij} \quad \forall (i, j) \in E \quad (2)$$

where

$$d(i, t) = \begin{cases} - \sum_{s \in N} D(s, t) & \text{if } i = t \\ D(i, t) & \text{if } i \neq t \end{cases}$$

The variables $y_{ij}^t$ denote the amount of traffic to $t$ routed through the link $(i, j)$. The equation set (1) state the condition that, at intermediate nodes $i$ (i.e., at nodes different from $t$), the outgoing traffic equals the incoming traffic plus traffic created at $i$ and destined to $t$, while at $t$ the incoming traffic equals all traffic destined to $t$. The equation set (2) state the condition that the total traffic routed over a link cannot exceed the link’s capacity.

It will also be of interest to consider the corresponding problem without requiring the presence of the equation set (2). We denote this problem:

$$\min \{ f(y) \mid y \in P_1 \}$$

(MCF$_1$)
where \( P_1 \) is the polyhedron defined by the set of equations (1). Notice that every point \( y = (y_{ij}^l) \) in \( P_{12} \) or \( P_1 \) represents a possible solution to the routing problem; it gives a way to route traffic over the network so that the demand is met and capacity limits are respected (when it belongs to \( P_{12} \)), or the demand is met but capacity limits are not necessarily respected (when it belongs to \( P_1 \)).

A general linear objective function for either problem has the form \( f(y) = \sum_{l,(i,j)} b_{ij}^l y_{ij}^l \). We will, however, consider only the case when all \( b_{ij}^l = 1 \) which corresponds to the case where all commodities have the same cost on all links.

3 Preliminaries

This section contains the definitions of efficient, \((\mathcal{L},\mathcal{E})\)-balanced solutions and some preliminary results. For the rest of the paper we simplify the notation and use \( e \) instead of \((i,j)\) to denote directed edges. The function considered above, \( f(y) = \sum_{l,e} y_{e}^l \), will be used as a measure of efficiency. We say that \( y_1 \) is more efficient than \( y_2 \) if \( f(y_1) \leq f(y_2) \), where \( y_1, y_2 \) belong to \( P_{12} \) or \( P_1 \). To motivate this definition, note that whenever traffic between two nodes can be routed over two different paths of unequal length, \( f \) will choose the shortest one. In case the capacity of the shortest path is not sufficient to send the requested traffic, \( f \) will utilise the shortest path to 100% of its capacity and send the remaining traffic over the longer path.

Given a point \( y = (y_{ij}^l) \) as above, we let \( Y_e = \sum_{l \in \mathcal{L}} y_{e}^l \) denote the total traffic sent through \( e \) by \( y \). Every such \( y \) defines a utilisation of edges by the formula

\[
u(y,e) = \begin{cases} \frac{\sum_{l \in \mathcal{L}} y_{e}^l}{k_e} = \frac{Y_e}{k_e} & \text{if } k_e > 0 \\ 0 & \text{if } k_e = 0 \end{cases}
\]

Recall that a partition \( \mathcal{E} = (E_1, \ldots, E_m) \) (of the set of edges \( E \)) is a collection of non-empty, pair-wise disjoint subsets of \( E \) whose union is \( E \). Given a partition \( \mathcal{E} \) and \( y \in P_{12} \) or \( P_1 \), utilisation is defined to be the \( m \)-dimensional vector:

\[
u(y,\mathcal{E}) = (\max_{e \in E_1}, \ldots, \max_{e \in E_m})
\]

Suppose that \( \mathcal{L} = (l_1, \ldots, l_m) \) is a vector of real numbers satisfying \( 0 < l_i < 1 \) for \( i = 1, \ldots, m \). We say that \( y \in P_{12} \) or \( P_1 \) is \((\mathcal{L},\mathcal{E})\)-balanced if \( \nu(y,\mathcal{E}) \leq \mathcal{L} \), where the inequality is to be understood component-wise.

Given a partition \( \mathcal{E} \), a sequence \( \mathcal{L} \) as above, and a real number \( \lambda > 1 \), define

\[
f^{\mathcal{L},\mathcal{E},\lambda}(y) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{l_i,\lambda}(\nu(y,e))
\]

where the link cost function \( C^{l_i,\lambda} \) (illustrated in Fig. 1) is defined by

\[
C^{l_i,\lambda}(U) = \begin{cases} U & \text{if } U \leq l_i \\ \lambda U + (1 - \lambda) l_i & \text{if } U \geq l_i \end{cases}
\]

The following lemma summarises some properties of \( f^{\mathcal{L},\mathcal{E},\lambda} \).
Lemma 1. Using the above notation, we have:
1. For all $y \in P_1$, $f(y) \leq f^{E,E,\lambda}(y)$.

2. If $y \in P_1$ is $(\mathcal{L},E)$-balanced, then $f^{E,E,\lambda}(y) = f(y)$.

Proof. 1) Since $C^{E,\lambda}(U) \geq U$ for all $U \geq 0$, we have:

$$f(y) = \sum_{e \in E} Y_e = \sum_{i=0}^{m} \sum_{e \in E_i} Y_e = \sum_{i=0}^{m} \sum_{e \in E_i} k_u(y,e)$$

$$\leq \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{E,\lambda}(u(y,e)) = f^{E,E,\lambda}(y).$$

2) Suppose that $y$ is $(\mathcal{L},E)$-balanced and $e \in E_i$. Then $u(y,e) \leq \ell_i$ and hence $C^{E,\lambda}(u(y,e)) = u(y,e)$. Thus

$$f^{E,E,\lambda}(y) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{E,\lambda}(u(y,e)) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e u(y,e) = f(y).$$

This completes the proof of the lemma.

Corollary 1. Suppose that $y_1, y_2 \in P_1$ are $(\mathcal{L},E)$-balanced and $y_1$ is optimal for $f^{E,E,\lambda}$. Then $f(y_1) \leq f(y_2)$.

Proof. Follows immediately from the assumptions and part 2) of the lemma.

4 The result

Before stating the theorem, we need to define a few constants. Let

$$v = \min \{ f(y) \mid y \in P_{12} \} \quad \text{and} \quad V = \max \{ f(y) \mid y \in P_{12} \}$$

Notice that $v > 0$ since $D(s,t) > 0$, and $V < \infty$ since the network is finite and we are enforcing the (finite) capacity conditions. Thus, $0 < v < V < \infty$.

Given $\mathcal{L} = (\ell_1, \ldots, \ell_m)$, $\mathcal{L} + \epsilon$ denotes $(\ell_1 + \epsilon, \ldots, \ell_m + \epsilon)$. Finally, let $\delta > 0$ denote the minimum capacity of the edges of positive capacity.

Theorem 1. Let $\mathcal{E}$ and $\mathcal{L}$ be as above, and let $\epsilon$ denote a real number satisfying $0 < \epsilon < \min_{1 \leq i \leq m}(1 - \ell_i)$. Suppose that $y \in P_1$ is $(\mathcal{L},\mathcal{E})$-balanced, and let $\lambda \geq 1 + \frac{v^2}{\epsilon^2}$. Then any solution $x$ of MCF with objective function $f^{\mathcal{E},\mathcal{E},\lambda}$ is $(\mathcal{L} + \epsilon,\mathcal{E})$-balanced. Moreover, $x$ is more efficient than any other $(\mathcal{L} + \epsilon,\mathcal{E})$-balanced point of $P_1$. 
Proof. Suppose that \( x \in P_1 \) is a solution of MCF1 with objective function \( f^{\mathcal{L},\mathcal{E},\lambda} \), for some \( \lambda \geq 1 + \frac{V^2}{e_{\max}} \), and let \( y \in P_1 \) be \((\mathcal{L}, \mathcal{E})\)-balanced. We claim that

\[
f^{\mathcal{L},\mathcal{E},\lambda}(x) \leq f^{\mathcal{L},\mathcal{E},\lambda}(y) = f(y) \leq \left( \frac{V}{v} \right) v \leq \left( \frac{V}{v} \right) f(x)
\]

so that, in particular,

\[
f^{\mathcal{L},\mathcal{E},\lambda}(x) \leq \left( \frac{V}{v} \right) f(x)
\]

Indeed, the first (in)equality in (3) is true because \( x \) is optimal for \( f^{\mathcal{L},\mathcal{E},\lambda} \), the second follows from (2) of the lemma, and the last two by the definitions of \( v \) and \( V \).

We will assume, for contradiction, that \( x \) is not \((\mathcal{L} + e, \mathcal{E})\)-balanced, i.e. that for some \( i \), \( 1 \leq i \leq m \), there is an edge \( e \in E_i \) such that \( u(x, e) > \ell_i + e \). Let \( E'_i = \{ e \in E_i \mid u(x, e) > \ell_i + e \} \) and note that, by assumption, \( E'_i \) is not empty. In (5) and (6) below we use the convenient notation:

\[
\sum_C \cdots = \sum_{j=0}^m \sum_{e \in E_j} \cdots - \sum_{e \in E'_j} \cdots
\]

Set \( X_e = \sum_{e \in E_j} x_e \). In (6) below we use the fact that \( C^{\mathcal{L},\lambda}(U) \geq U \):

\[
f^{\mathcal{L},\mathcal{E},\lambda}(x) = \sum_C k_C C^{\mathcal{L},\lambda}(u(y, e)) + \sum_{e \in E'_j} k_e C^{\mathcal{L},\lambda}(u(y, e)) \geq \sum_{C} k_C u(y, e) + \sum_{e \in E'_j} k_e (C^{\mathcal{L},\lambda}(u(y, e)) - u(x, e))
\]

\[
= \sum_{j=0}^m \sum_{e \in E_j} X_e + \sum_{e \in E'_j} k_e (C^{\mathcal{L},\lambda}(u(y, e)) - u(x, e))
\]

\[
= f(x) + \sum_{e \in E'_j} k_e ((\lambda - 1)u(y, e) + (1 - \lambda)\ell_i)
\]

It follows from the inequality we have just obtained, together with (4), that

\[
(\lambda - 1) \sum_{e \in E'_j} (X_e - k_e \ell_i) \leq \left( \frac{V}{v} - 1 \right) f(x).
\]

But, taking into account the fact that \( \lambda > 1 \), we obtain

\[
(\lambda - 1) \sum_{e \in E'_j} (X_e - k_e \ell_i) = (\lambda - 1) \sum_{e \in E'_j} k_e (u(x, e) - \ell_i) > (\lambda - 1) e \sum_{e \in E'_j} k_e
\]

\[
\geq (\lambda - 1) e \delta \geq \frac{V^2}{v} \geq \left( \frac{V}{v} \right) f(x)
\]
contradicting (7). The last assertion of the theorem follows directly from the corollary. This completes the proof of Theorem 1.

Corollary 2 below is the special case when \( m = 1 \), i.e. \( \mathcal{E} = (E) \) and \( \mathcal{L} = (L) \). It was formulated without proof as Theorem 1 of [1]. In this special case we simplify the notation and simply write \( f^{L,E,\lambda} \) instead of \( f^{L,E,\lambda} \).

**Corollary 2.** Let \( \ell, \epsilon \) be real numbers satisfying \( 0 < \ell < 1 \) and \( 0 < \epsilon < 1 - \ell \). Suppose that \( y \in P_1 \) is \( \ell \)-balanced, and let \( \lambda \geq 1 + \frac{1}{\ell(1-\epsilon)} \). Then any solution \( x \) of \( (MCF_1) \) with objective function \( f^{L,E,\lambda} \) is \((\ell + \epsilon)\)-balanced. Moreover, \( x \) is more efficient than any other \((\ell + \epsilon)\)-balanced point of \( P_1 \).

The attentive reader may have wondered why, if we are interested in finding efficient \((\mathcal{L},\mathcal{E})\)-balanced solutions, we have not used the following more direct approach. Consider the problem

\[
\min \{ f(y) \mid y \in P_{1,L} \} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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