SICS Technical Report
T2002:05

ISRN: SICS-T–2002/05-SE
ISSN: 1100-3154

Objective Functions for Balance in Traffic Engineering

Juan Alonso, Henrik Abrahamsson, Bengt Ahlgren,
Anders Andersson and Per Kreuger

{alonso,henrik,bengta,andersa,piak}@sics.se

May 2002

Swedish Institute of Computer Science
Box 1263, SE-164 29 Kista, Sweden

Abstract. We prove a result concerning objective functions that can be used to obtain *efficient* and *balanced* solutions to the multi-commodity network flow problem. This type of solution is of interest when routing traffic in the Internet. A particular case of the result proved here (see Corollary 2 below) was stated without proof in a previous paper.

Objective Functions for Balance in Traffic Engineering*

Juan Alonso, Henrik Abrahamsson, Bengt Ahlgren, Anders Andersson, and Per Kreuger

SICS – Swedish Institute of Computer Science
E-mail: `first.lastname@sics.se`

Abstract. We prove a result concerning objective functions that can be used to obtain *efficient* and *balanced* solutions to the multi-commodity network flow problem. This type of solution is of interest when routing traffic in the Internet. A particular case of the result proved here (see Corollary 2 below) was stated without proof in a previous paper.

1 Introduction

Traffic engineering encompasses performance evaluation and performance optimisation of operational IP networks. An important goal with traffic engineering is to use the available network resources more efficiently for different types of load patterns and to avoid congestion by having a relatively balanced distribution of traffic over the network.

Current routing protocols in the Internet calculate the shortest path to a destination in some metric without knowing anything about the traffic demand. Manual configuration by the network operator is therefore necessary to balance load between available paths to avoid congestion. One way of simplifying the task of the operator and improve use of the available network resources is to make the routing protocol sensitive to traffic demand. Routing then becomes a flow optimisation problem.

In another paper [1] we discussed a new routing algorithm based on multi-commodity flow optimisation. In this report we present and prove a theorem that has, as a special case, a result stated without proof in the mentioned paper. The theorem concerns optimisation objective functions which allow the network operator to choose a maximum desired link utilisation level. The optimisation will then find the most efficient solution, if it exists, satisfying this constraint. The objective function thus enables the operator to control the trade-off between minimising the network utilisation and balancing load over multiple paths.

We have tried to make the paper self-contained by giving enough background information. The interested reader is referred to a previous paper [1] and the references quoted there for more information on IP networks and on optimisation.

The rest of this report is organised as follows. Section 2 recalls the formulation of the multi-commodity flow problem as given in an earlier paper [1]. Section 3

* Supported in part by Telia Research AB.

defines efficiency, balance and the objective functions of interest. The theorem and its proof are presented in Section 4.

2 Formulation of the multi-commodity flow problem

The routing problem in a network consists in finding a path or multiple paths that send the requested traffic through the network without exceeding the capacity of the links. In a previous paper [1] we modelled the routing problem as a multi-commodity network flow problem (MCF) as follows.

We represent the network by a directed graph $G = (N, E)$, where N is a set of nodes and E is a set of (directed) edges. We assume the graph is such that its edges can be uniquely represented by an ordered pair (i, j) of nodes, where i is the initial point of the edge and j its final point. Every edge $(i, j) \in E$ has an associated *capacity* k_{ij} reflecting the bandwidth available to the corresponding link. In addition, we assume given a *demand matrix* $D = D(s, t)$ expressing the traffic demand from node s to node t in the network. The entries of the demand matrix are non-negative and, to avoid trivialities, we assume that $D(s, t) > 0$ for at least one pair of nodes. We model commodities as (only destination) nodes, i.e., a commodity t is to be interpreted as "all traffic to t". Then the corresponding (MCF) problem can be formulated as follows:

$$\min \{f(y) \,|\, y \in P_{12}\} \qquad (MCF_{12})$$

where $y = (y^t_{ij})$, for $t \in N, (i, j) \in E$, and P_{12} is the polyhedron defined by the equations:

$$\sum_{\{j|(i,j)\in E\}} y^t_{ij} - \sum_{\{j|(j,i)\in E\}} y^t_{ji} = d(i,t) \qquad \forall i, t \in N \qquad (1)$$

$$\sum_{t\in N} y^t_{ij} \leq k_{ij} \qquad \forall (i,j) \in E \qquad (2)$$

where

$$d(i,t) = \begin{cases} -\sum_{s\in N} D(s,t) & \text{if } i = t \\ D(i,t) & \text{if } i \neq t \end{cases}$$

The variables y^t_{ij} denote the amount of traffic to t routed through the link (i, j). The equation set (1) state the condition that, at intermediate nodes i (i.e., at nodes different from t), the outgoing traffic equals the incoming traffic plus traffic created at i and destined to t, while at t the incoming traffic equals all traffic destined to t. The equation set (2) state the condition that the total traffic routed over a link cannot exceed the link's capacity.

It will also be of interest to consider the corresponding problem *without* requiring the presence of the equation set (2). We denote this problem :

$$\min \{f(y) \,|\, y \in P_1\} \qquad (MCF_1)$$

where P_1 is the polyhedron defined by the set of equations (1). Notice that every point $y = (y_{ij}^t)$ in P_{12} or P_1 represents a possible solution to the routing problem: it gives a way to route traffic over the network so that the demand is met and capacity limits are respected (when it belongs to P_{12}), or the demand is met but capacity limits are not necessarily respected (when it belongs to P_1).

A general linear objective function for either problem has the form $f(y) = \sum_{t,(i,j)} b_{ij}^t\, y_{ij}^t$. We will, however, consider only the case when all $b_{ij}^t = 1$ which corresponds to the case where all commodities have the same cost on all links.

3 Preliminaries

This section contains the definitions of *efficient*, $(\mathcal{L}, \mathcal{E})$-*balanced* solutions and some preliminary results. For the rest of the paper we simplify the notation and use e instead of (i, j) to denote directed edges . The function considered above, $f(y) = \sum_{t,e} y_e^t$, will be used as a measure of efficiency. We say that y_1 is *more efficient* than y_2 if $f(y_1) \leq f(y_2)$, where y_1, y_2 belong to P_{12} or P_1. To motivate this definition, note that whenever traffic between two nodes can be routed over two different paths of unequal length, f will choose the shortest one. In case the capacity of the shortest path is not sufficient to send the requested traffic, f will utilise the shortest path to 100% of its capacity and send the remaining traffic over the longer path.

Given a point $y = (y_e^t)$ as above, we let $Y_e = \sum_{t \in N} y_e^t$ denote the total traffic sent through e by y. Every such y defines a *utilisation* of edges by the formula

$$u(y, e) = \begin{cases} \dfrac{\sum_{t \in N} y_e^t}{k_e} = \dfrac{Y_e}{k_e} & \text{if } k_e > 0 \\ 0 & \text{if } k_e = 0 \end{cases}$$

Recall that a *partition* $\mathcal{E} = (E_1, \ldots, E_m)$ (of the set of edges E) is a collection of non-empty, pair-wise disjoint subsets of E whose union is E. Given a partition $\mathcal{E}$ and $y \in P_{12}$ or P_1, *utilisation* is defined to be the m-dimensional vector:

$$u(y, \mathcal{E}) = (\max_{e \in E_1}, \ldots, \max_{e \in E_m})$$

Suppose that $\mathcal{L} = (\ell_1, \ldots, \ell_m)$ is a vector of real numbers satisfying $0 < \ell_i < 1$ for $i = 1, \ldots, m$. We say that $y \in P_{12}$ or P_1 is $(\mathcal{L}, \mathcal{E})$-*balanced* if $u(y, \mathcal{E}) \leq \mathcal{L}$, where the inequality is to be understood component-wise.

Given a partition $\mathcal{E}$, a sequence $\mathcal{L}$ as above, and a real number $\lambda > 1$, define

$$f^{\mathcal{L},\mathcal{E},\lambda}(y) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{\ell_i,\lambda}\,(u(y, e))$$

where the *link cost function* $C^{\ell_i,\lambda}$ (illustrated in Fig. 1) is defined by

$$C^{\ell_i,\lambda}(U) = \begin{cases} U & \text{if } U \leq \ell_i \\ \lambda\, U + (1 - \lambda)\, \ell_i & \text{if } U \geq \ell_i \end{cases}$$

The following lemma summarises some properties of $f^{\mathcal{L},\mathcal{E},\lambda}$.

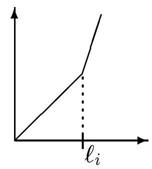

Fig. 1. The link cost function $C^{\ell_i,\lambda}$.

Lemma 1. *Using the above notation, we have:*

1. *For all $y \in P_1$, $f(y) \leq f^{\mathcal{L},\mathcal{E},\lambda}(y)$.*
2. *If $y \in P_1$ is $(\mathcal{L},\mathcal{E})$-balanced, then $f^{\mathcal{L},\mathcal{E},\lambda}(y) = f(y)$.*

Proof. 1) Since $C^{\ell_i,\lambda}(U) \geq U$ for all $U \geq 0$, we have:

$$\begin{aligned} f(y) &= \sum_{e \in E} Y_e = \sum_{i=0}^{m} \sum_{e \in E_i} Y_e = \sum_{i=0}^{m} \sum_{e \in E_i} k_e u(y,e) \\ &\leq \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{\ell_i,\lambda}(u(y,e)) = f^{\mathcal{L},\mathcal{E},\lambda}(y). \end{aligned}$$

2) Suppose that y is $(\mathcal{L},\mathcal{E})$-balanced and $e \in E_i$. Then $u(y,e) \leq \ell_i$ and hence $C^{\ell_i,\lambda}(u(y,e)) = u(y,e)$. Thus

$$f^{\mathcal{L},\mathcal{E},\lambda}(y) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e C^{\ell_i,\lambda}(u(y,e)) = \sum_{i=0}^{m} \sum_{e \in E_i} k_e u(y,e) = f(y).$$

This completes the proof of the lemma.

Corollary 1. *Suppose that y_1, $y_2 \in P_1$ are $(\mathcal{L},\mathcal{E})$-balanced and y_1 is optimal for $f^{\mathcal{L},\mathcal{E},\lambda}$. Then $f(y_1) \leq f(y_2)$.*

Proof. Follows immediately from the assumptions and part 2) of the lemma.

4 The result

Before stating the theorem, we need to define a few constants. Let

$$v = \min\{f(y) \,|\, y \in P_{12}\} \qquad \text{and} \qquad V = \max\{f(y) \,|\, y \in P_{12}\}$$

Notice that $v > 0$ since $D(s,t) > 0$, and $V < \infty$ since the network is finite and we are enforcing the (finite) capacity conditions. Thus, $0 < v \leq V < \infty$.

Given $\mathcal{L} = (\ell_1, \ldots, \ell_m)$, $\mathcal{L} + \epsilon$ denotes $(\ell_1 + \epsilon, \ldots, \ell_m + \epsilon)$. Finally, let $\delta > 0$ denote the minimum capacity of the edges of positive capacity.

Theorem 1. *Let $\mathcal{E}$ and $\mathcal{L}$ be as above, and let ϵ denote a real number satisfying $0 < \epsilon < \min_{1 \leq i \leq m}(1 - \ell_i)$. Suppose that $y \in P_1$ is $(\mathcal{L},\mathcal{E})$-balanced, and let $\lambda \geq 1 + \frac{V^2}{v\delta\epsilon}$. Then any solution x of MCF_1 with objective function $f^{\mathcal{L},\mathcal{E},\lambda}$ is $(\mathcal{L}+\epsilon,\mathcal{E})$-balanced. Moreover, x is more efficient than any other $(\mathcal{L}+\epsilon,\mathcal{E})$-balanced point of P_1.*

Proof. Suppose that $x \in P_1$ is a solution of MCF_1 with objective function $f^{\mathcal{L},\mathcal{E},\lambda}$, for some $\lambda \geq 1 + \frac{V^2}{v\delta\epsilon}$, and let $y \in P_1$ be $(\mathcal{L},\mathcal{E})$-balanced. We claim that

$$f^{\mathcal{L},\mathcal{E},\lambda}(x) \leq f^{\mathcal{L},\mathcal{E},\lambda}(y) = f(y) \leq \left(\frac{V}{v}\right) v \leq \left(\frac{V}{v}\right) f(x) \tag{3}$$

so that, in particular,

$$f^{\mathcal{L},\mathcal{E},\lambda}(x) \leq \left(\frac{V}{v}\right) f(x) \tag{4}$$

Indeed, the first (in)equality in (3) is true because x is optimal for $f^{\mathcal{L},\mathcal{E},\lambda}$, the second follows from 2) of the lemma, and the last two by the definitions of v and V.

We will assume, for contradiction, that x is not $(\mathcal{L}+\epsilon,\mathcal{E})$-balanced, i.e. that for some i, $1 \leq i \leq m$, there is an edge $e \in E_i$ such that $u(x,e) > \ell_i + \epsilon$. Let $E_i' = \{e \in E_i | u(x,e) > \ell_i + \epsilon\}$ and note that, by assumption, E_i' is not empty. In (5) and (6) below we use the convenient notation:

$$\sum_{C} \cdots = \sum_{j=0}^{m} \sum_{e \in E_j} \cdots - \sum_{e \in E_j'} \cdots$$

Set $X_e = \sum_{t \in N} x_e^t$. In (6) below we use the fact that $C^{\ell_i,\lambda}(U) \geq U$:

$$\begin{aligned} f^{\mathcal{L},\mathcal{E},\lambda}(x) &= \sum_{C} k_e C^{\ell_i,\lambda}\left(u(y,e)\right) + \sum_{e \in E_i'} k_e C^{\ell_i,\lambda}\left(u(y,e)\right) && (5)\\ &\geq \sum_{C} k_e u(y,e) + \sum_{e \in E_i'} k_e C^{\ell_i,\lambda}\left(u(y,e)\right) && (6)\\ &= \sum_{j=0}^{m} \sum_{e \in E_j} X_e + \sum_{e \in E_i'} k_e \left(C^{\ell_i,\lambda}\left(u(y,e)\right) - u(x,e)\right) \\ &= f(x) + \sum_{e \in E_i'} k_e \left((\lambda-1)u(y,e) + (1-\lambda)\ell_i\right) \\ &= f(x) + (\lambda-1) \sum_{e \in E_i'} (X_e - k_e \ell_i) \end{aligned}$$

It follows from the inequality we have just obtained, together with (4), that

$$(\lambda-1) \sum_{e \in E_i'} (X_e - k_e \ell_i) \leq \left(\frac{V}{v} - 1\right) f(x). \tag{7}$$

But, taking into account the fact that $\lambda > 1$, we obtain

$$\begin{aligned} (\lambda-1) \sum_{e \in E_i'} (X_e - k_e \ell_i) &= (\lambda-1) \sum_{e \in E_i'} k_e (u(x,e) - \ell_i) > (\lambda-1)\epsilon \sum_{e \in E_i'} k_e \\ &\geq (\lambda-1)\epsilon\delta \geq \frac{V^2}{v} \geq \left(\frac{V}{v}\right) f(x) \end{aligned}$$

contradicting (7). The last assertion of the theorem follows directly from the corollary. This completes the proof of Theorem 1.

Corollary 2 below is the special case when $m = 1$, i.e. $\mathcal{E} = (E)$ and $\mathcal{L} = (\ell)$. It was formulated without proof as Theorem 1 of [1]. In this special case we simplify the notation and simply write $f^{\ell,\lambda}$ instead of $f^{\mathcal{L},\mathcal{E},\lambda}$.

Corollary 2. *Let ℓ, ϵ be real numbers satisfying $0 < \ell < 1$ and $0 < \epsilon < 1 - \ell$. Suppose that $y \in P_1$ is ℓ-balanced, and let $\lambda \geq 1 + \frac{V^2}{v\delta\epsilon}$. Then any solution x of MCF_1 with objective function $f^{\ell,\lambda}$ is $(\ell + \epsilon)$-balanced. Moreover, x is more efficient than any other $(\ell + \epsilon)$-balanced point of P_1.*

The attentive reader may have wondered why, if we are interested in finding efficient $(\mathcal{L}, \mathcal{E})$-balanced solutions, we have not used the following more direct approach. Consider the problem

$$\min \{f(y) \,|\, y \in P_{1\mathcal{L}}\} \qquad\qquad (MCF_{1\mathcal{L}})$$

where $P_{1\mathcal{L}}$ denotes the polyhedron defined by the equation set (1) together with the following equations

$$\sum_{t \in N} y_e^t \;\leq\; \ell_i k_e \qquad \forall i (1 \leq i \leq m) \; \forall e \in E_i$$

When $(\mathcal{L}, \mathcal{E})$-balanced solutions exist, solving $(MCF_{1\mathcal{L}})$ will produce efficient $(\mathcal{L}, \mathcal{E})$-balanced solutions, and the method of Theorem 1 will produce $(\mathcal{L} + \epsilon, \mathcal{E})$-balanced solutions. Since we can choose ϵ arbitrarily small, the two methods are essentially equivalent. When no $(\mathcal{L}, \mathcal{E})$-balanced solutions exist, however, the methods differ markedly. In this case $(MCF_{1\mathcal{L}})$ yields only the information that the problem is infeasible, whereas the method of Theorem 1 will produce a "best effort" solution (which will of course not be $(\mathcal{L}, \mathcal{E})$-balanced). We call the solution "best effort" because $f^{\mathcal{L},\mathcal{E},\lambda}$, by penalising edges with high utilisation, gives preference to solutions that are as "balanced" as possible. Given that the application we have in mind is routing traffic in the Internet (see [1]), and that time is important, it should be clear that the method proposed in Theorem 1 offers a considerable practical advantage over the alternative provided by $(MCF_{1\mathcal{L}})$.

References

[1] H. Abrahamsson, B. Ahlgren, J. Alonso, A. Andersson, and P. Kreuger. A Multi Path Routing Algorithm for IP Networks Based on Flow Optimisation Submitted to QofIS, May 2002