In Proc. Reglermöte 2006, May 30-31, 2006.
KTH, Stockholm, Sweden.

Kalman Filtering with Unknown Noise Covariances

Martin Nilsson

Swedish Institute of Computer Science,
POB 1263, 164 29 Kista
E-mail: from.reglermote2006 AT drnil DOT com

Abstract: Since it is often difficult to identify the noise covariances for a Kalman filter, they are commonly considered design variables. If so, we can as well try to choose them so that the corresponding Kalman filter has some nice form. In this paper, we introduce a one-parameter subfamily of Kalman filters with the property that the covariance parameters cancel in the expression for the Kalman gain. We provide a simple criterion which guarantees that the implicitly defined process covariance matrix is positive definite.

Key words and phrases: *discrete-time linear system, process noise, measurement noise, covariance, Kalman filter, discrete Riccati equation, singular value decomposition, Moore-Penrose pseudoinverse.*

1. INTRODUCTION

We consider discrete-time linear systems expressed in the form

$$\begin{aligned} x_{k+1} &= F x_k + M w_k && (1) \\ y_k &= H x_k + N v_k && (2) \end{aligned}$$

where $\{w_k\}$ and $\{v_k\}$ are uncorrelated sequences of white noise with unit intensity. F and H are known matrices with (F, H) completely observable, F non-singular, and y_k are measurements. We want to estimate the n-dimensional hidden state vector $x_k \approx \hat{x}_k$, by introducing a Kalman filter

$$\hat{x}_{k+1} = F\,\hat{x}_k + K\,(y_k - H\,\hat{x}_k) \qquad (3)$$

where K is the Kalman gain

$$K = FPH^T(HPH^T + R)^{-1} \qquad (4)$$

and where $R = N\ N^T$ is the measurement noise covariance, $P = E[(x - \hat{x})^2]$ is the error covariance, and $Q = M\ M^T$ is the process noise covariance. P, Q, and R are related through the discrete-time Riccati equation

$$\begin{aligned} P &= Q + FPF^T \\ &\quad - FPH^T\,(HPH^T + R)^{-1}\,H^T PF^T \end{aligned} \qquad (5)$$

R might be estimated by making measurements and calculating the variance, but estimating Q is more difficult, since the state vector x cannot be measured directly. Also, Q acts as a "waste basket" for unknown modelling errors. Many methods for estimating R and Q from the output sequence $\{y_k\}$ have been proposed. Overviews of such methods can be found in e.g. [1,5]. Some of the methods (Bayesian, maximum likelihood, time series, correlation, and subspace methods) require considerable computing time and memory. For adaptive systems where covariances need to be estimated on-line, covariance matching methods [6-8] have become popular due to their simplicity and speed, despite being suboptimal.

The origin of this paper is an attempt to introduce Kalman filtering to students on novice level in the simplest possible way. Although there is an abundance of material on the Kalman filter, and a large number of research reports on methods for estimating noise covariances, the elementary literature is sparse on the subject. In practice, Q and R are often considered design variables [2,3,9], and chosen *ad hoc*. A common approach to choose the covariances is Bryson's rule [9], where Q is chosen as a diagonal weight matrix.

In this paper. we propose a method based on the idea of using the discrete Riccati equation backwards: If Q is considered a design variable anyway, we use the equation to estimate Q from P instead of the other way around. A difficulty when using this approach is to guarantee the positive-definiteness of Q [4], but in our case, a concise criterion can be derived easily.

We arrive at a simple expression for the filter, which doesn't contain any explicit references to R or Q.

1.1 *Notation and terminology*

We will use instances of *singular value decomposition* [10]. Any $m \times n$ matrix H of rank r can be uniquely written as

$$H = U\,\Sigma\,V^T \qquad (6)$$

where U and V are orthonormal matrices and Σ is an $m \times n$ matrix where the first r diagonal elements, the *singular values*, d_i, $i = 1...r$, are the only non-zero elements, positive, and in decreasing order. The largest singular value is $d_i = \bar{\sigma}_H$ and the smallest singular value is $d_r = \underline{\sigma}_H$. The *norm* of H is $\left\| H \right\| = \bar{\sigma}_H$ (in a generalized sense, since H is not necessarily square). We define the *condition number* of H as $\kappa_H = \bar{\sigma}_H / \underline{\sigma}_H$. The *Moore-Penrose pseudoinverse* H^+ of H is defined as

$$H^+ = V\,\Sigma^+ U^T \tag{7}$$

where the diagonal elements of Σ^+ are $d_i^+ = 1/d_i$. Some useful properties of the pseudoinverse are that $H = H\;H^+\;H$, $H^+ = H^+\;H\;H^+$, and $H^+\;H = (H^+\,H)^T$. For any positive definite matrix R,

$$\underline{\sigma}_R\,|x|^2 \le x^T R\,x \le \bar{\sigma}_R\,|x|^2 \tag{8}$$

The bounds in this inequality are tight.

2. DERIVATION OF THE FILTER

Vaguely expressed, if we choose too small a Q, the Kalman filter will converge too slowly, but if we make Q too large, then P and K will also become large, and the filter becomes overly sensitive. How large a Q is acceptable? A very rough idea is to make Q so large that it just about matches the effects of the measurement noise R. The Riccati equation leads us to the guess that this happens when

$$H\,P H^T \approx c\,R \tag{1}$$

where c is a scalar positive tuning factor. P and R are covariance matrices and thus must be symmetric and positive semidefinite. A choice which makes P a symmetric, positive semidefinite matrix is

$$P = c\,H^+\,R\,(H^T)^+ = c\,H^+\,R\,(H^+)^T \tag{2}$$

We are interested in making P small, and an attractive property of the pseudoinverse is that $x = H^+\,b$ is the least squares solution to the equation $H x = b$. When the expression for P is inserted into the Riccati equation (5, section 1) we obtain an expression for Q. A complication here is that we must also ensure that Q is positive semidefinite.

We require that H is full column-rank. If not, we can transform the system in the following way. Since we required the original system (1-2) in section 1 to be completely observable, we can add old measurements to the list of outputs, extending the output matrix H to the observability matrix. We may add more old outputs if we want to improve on the ill-conditioned problem of directly inverting the observability matrix. The new system becomes

$$x_{k+1} = F x_k + M w_k \tag{3}$$

$$y'_k = \begin{pmatrix} H \\ H\,F^{-1} \\ \vdots \\ H\,F^{-p+1} \end{pmatrix} x_k + N'\,v'_k \tag{4}$$

where

$$y'_k = \begin{pmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-p+1} \end{pmatrix} \tag{5}$$

which is of the same form as (1-2) in section 1, except that the noise sequence $\{v'_k\}$ is now correlated.

We have

$$\begin{aligned} K &= F P H^T (H P H^T + R)^{-1} \\ &= \frac{c}{1+c} F H^+ \end{aligned} \tag{6}$$

The filter equation can be written

$$\begin{aligned} \hat{x}_{k+1} &= F\hat{x}_k + K\,(y_k - H\,\hat{x}_k) \\ &= F\hat{x}_k + \frac{c}{c+1} F H^+ (y_k - H\,\hat{x}_k) \\ &= F\,\frac{\hat{x}_k + c\,H^+\,y_k}{1+c} \end{aligned} \tag{7}$$

Reconstructing x by forming $H^+ y$ is equivalent to finding x from y by least squares. The stability of the scheme can be seen from the relation

$$\begin{aligned} x_k - \hat{x}_k &= (1-\theta) F (x_{k-1} - \hat{x}_{k-1}) \\ &\quad + (M w_{k-1} - (1-\theta)\,F H^+ N v_{k-1}) \end{aligned} \tag{8}$$

where $\theta = 1/(c+1)$, demonstrating the scheme to be stable when $(1-\theta)\|F\| < 1$.

3. THE PROCESS NOISE COVARIANCE

We must now ensure that Q is positive definite.

$$\begin{aligned} Q &= P - F P F^T + K\,H\,P F^T \\ &= P - F P F^T + \frac{c}{1+c} F H^+ H P F^T \\ &= P - \frac{1}{1+c} F P F^T \end{aligned} \tag{1}$$

Since

$$\begin{aligned} x^T P x &= c\,(x^T H^+)\,R\,(x^T H^+)^T \\ &\ge c|x|^2 \frac{\underline{\sigma}_R}{\bar{\sigma}_H^2} \end{aligned} \tag{2}$$

and

$$x^T\,FPF^T x \le c\,|x|^2 \bar{\sigma}_F^2 \frac{\bar{\sigma}_R}{\underline{\sigma}_H^2} \tag{3}$$

we have

$$x^T Q\,x \ge c\,|x|^2 \left(\frac{\underline{\sigma}_R}{\bar{\sigma}_H^2} - \frac{\bar{\sigma}_F^2}{c+1} \frac{\bar{\sigma}_R}{\underline{\sigma}_H^2} \right) \tag{4}$$

Q is surely positive definite when this expression is positive, which happens when

$$\frac{1}{1+c} < \frac{1}{\bar{\sigma}_F^2 \kappa_H^2 \kappa_R} \tag{5}$$

In the same way as above,

$$x^T Q\,x \le c\,|x|^2 \left(\frac{\bar{\sigma}_R}{\underline{\sigma}_H^2} - \frac{\underline{\sigma}_F^2}{c+1} \frac{\underline{\sigma}_R}{\bar{\sigma}_H^2} \right) < c\,|x|^2 \frac{\bar{\sigma}_R}{\underline{\sigma}_H^2} \tag{6}$$

Since $x^T \underline{\sigma}_Q\,x$ and $x^T \bar{\sigma}_Q\,x$ are tight bounds for $x^T Q\,x$,

$$\frac{\underline{\sigma}_Q\,\underline{\sigma}_H^2}{\bar{\sigma}_R} < c \le \frac{\bar{\sigma}_Q\,\bar{\sigma}_H^2}{\underline{\sigma}_R} \left(1 - \frac{\bar{\sigma}_F^2}{c+1} \kappa_H^2 \kappa_R \right)^{-1} \tag{7}$$

If the ratio $\|Q\|/\|R\|$ is known, this inequality can be used as a basis for a guess at c,

$$c \approx \frac{\|Q\|\,\|H\|^2}{\|R\|} \tag{8}$$

4. AN EXAMPLE

Consider a case where we measure the position of an object and want to determine its speed. The system can be approximated

$$\begin{aligned} \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} &= \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} + M\,w_k && (1) \\ y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} + N\,v_k && (2) \end{aligned}$$

Since

$$\begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1 \end{pmatrix} \tag{3}$$

Augmenting the system by the three previous measurements,

$$\begin{pmatrix} y_k \\ y_{k-1} \\ y_{k-2} \\ y_{k-3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -\Delta t \\ 1 & -2\Delta t \\ 1 & -3\Delta t \end{pmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} + N'\,v'_k \tag{4}$$

we can write

$$\begin{aligned} H^+ &= (H^T H)^{-1} H^T && (5) \\ &= \begin{pmatrix} 0.7 & 0.4 & 0.1 & -0.2 \\ 0.3/\Delta t & 0.1/\Delta t & -0.1/\Delta t & -0.3/\Delta t \end{pmatrix} \end{aligned}$$

5. DISCUSSION AND CONCLUSIONS

We conclude that for some Q and R the intuitive observer

$$\hat{x}_{k+1} = F\left[\theta\;\hat{x}_k + (1-\theta)\;H^+\,y_k\right] \tag{1}$$

is a special case of a Kalman filter, provided the original H is full column-rank. The condition $(1-\theta)\|F\| < 1$ guarantees stability. The filter can be described as a weighted average of the old state and a least squares reconstruction from a set of recent measurements. Given any positive definite measurement noise covariance matrix R, the choice

$$\theta < \frac{1}{\|F\|^2\,\kappa_H^2\,\kappa_R} \tag{2}$$

where κ_H and κ_R are the condition numbers for H and R, guarantees that an implicitly defined matrix Q is positive definite.

The filter is usually suboptimal, of course, but can provide a starting point for further improvement. The filter becomes better, the closer the covariance matrices are to proportionality. It resembles an unreduced Luenberger observer, but doesn't use pole placement.

6. REFERENCES

[1] Mehra, R.K.: *Approaches to adaptive filtering.* IEEE Trans. Automatic Control. October 1972. pp. 693-698.

[2] Glad, T., Ljung, L.: *Reglerteori.* 2nd. ed. Studentlitteratur, 2003. ISBN 91-44-03003-7. p. 268.

[3] Gustafsson, F.: *Adaptive filtering and change detection.* John Wiley, 2000. ISBN 0-471-49287-6. p 15.

[4] Johansson, R., Verhaegen, M., Chou, C.T., Robertsson, A.: *Residual models and stochastic realization in state-space identification.* Int. J. Control, 2001. Vol. 74, No. 10. pp. 988-995.

[5] Odelson, B.J.: *Estimating Disturbance Covariances From Data For Improved Control Performance*. Ph.D. thesis, Dept. of Chemical Engineering, University of Wisconsin-Madison, 2003.

[6] Myers, K., Tapley, B.: *Adaptive sequential estimation with unknown noise statistics*. IEEE Trans. Automatic Control, 21:520-523, 1976.

[7] Sage, A., Husa, G.: *Adaptive filtering with unknown prior statistics*. In Proc. Joint Automat. Control Conf., Bould, CO, 1969. pp 760-769.

[8] Maybeck, P.S.: *Stochastic models, estimation and control*, Vol. 2, 1979.

[9] Bryson, A.E., Ho, Y.-C.: *Applied Optimal Control: optimization, estimation, and control*. New York, Hemisphere, 1975.

[10] Ben-Israel, A.E., Greville, T.N.E.: *Generalized Inverses: Theory and Applications*. 2nd ed. Springer, 2003.